$\Box_{[l_1,u_1]}\Box_{[l_2,u_2]}\varphi \mapsto \Box_{[l_1+l_2,u_1+u_2]}\varphi$	$\Diamond_{[l_1,u_1]}\Diamond_{[l_2,u_2]}\varphi \mapsto \Diamond_{[l_1+l_2,u_1+u_2]}\varphi$	(R1)
$\Box_{[l_1,u_1]}\varphi \wedge \Box_{[l_2,u_2]}\psi \mapsto \Box_{[l_3,u_3]}(\Box_{[l_1-l_3,u_1-u_3]}\varphi \wedge \Box_{[l_2-l_3,u_2-u_3]}\psi)$ $\Diamond_{[l_1,u_1]}\varphi \vee \Diamond_{[l_2,u_2]}\psi \mapsto \Diamond_{[l_3,u_3]}(\Diamond_{[l_1-l_3,u_1-u_3]}\varphi \vee \Diamond_{[l_2-l_3,u_2-u_3]}\psi)$ where $l_3 = min(l_1,l_2), u_3 = l_3 + min(u_1-l_1, u_2-l_2), l_3 < u_3$		(R2)
$\Box_{[a,a]}\Diamond_{[l,u]}\varphi \mapsto \Diamond_{[l+a,u+a]}\varphi$ $\Diamond_{[a,a]}\Box_{[l,u]}\varphi \mapsto \Box_{[l+a,u+a]}\varphi$	$\Diamond_{[l,u]}\Box_{[a,a]}\varphi \mapsto \Diamond_{[l+a,u+a]}\varphi$ $\Box_{[l,u]}\Diamond_{[a,a]}\varphi \mapsto \Box_{[l+a,u+a]}\varphi$	(R3)
$\Box_{[l_1,u_1]}\varphi \wedge \Box_{[l_2,u_2]}\varphi \mapsto \Box_{[l_1,u_3]}\varphi$ where $l_1 \leqslant u_1, l_2 \leqslant u_2, l_1 \leqslant l_2 \leqslant u_1 + 1, u_3 = max(u_1,u_2)$	$\Diamond_{[l_1,u_1]}\varphi \vee \Diamond_{[l_2,u_2]}\varphi \mapsto \Diamond_{[l_1,u_2]}\varphi$	(R4)
$\Box_{[l_1,u_1]}\varphi \vee \Box_{[l_2,u_2]}\varphi \mapsto \Box_{[l_2,u_2]}\varphi$ where $l_1 \leqslant l_2 \leqslant u_2 \leqslant u_1$	$\Diamond_{[l_1,u_1]}\varphi \wedge \Diamond_{[l_2,u_2]}\varphi \mapsto \Diamond_{[l_2,u_2]}\varphi$	(R5)
$\Box_{[a,a]}(\varphi\, \mathcal{U}_{[l,u]}\, \psi) \mapsto \varphi\, \mathcal{U}_{[l+a,u+a]}\, \psi$	$(\Box_{[a,a]}\varphi)\, \mathcal{U}_{[l,u]}\, (\Box_{[a,a]}\psi) \mapsto \varphi\, \mathcal{U}_{[l+a,u+a]}\, \psi$	(R6)
$(\varphi_1\, \mathcal{U}_{[l,u_1]}\, \varphi_2) \wedge (\varphi_3\, \mathcal{U}_{[l,u_2]}\, \varphi_2) \mapsto (\varphi_1 \wedge \varphi_3)\, \mathcal{U}_{[l,u_1]}\, \varphi_2$ where $l \leqslant u_1, l \leqslant u_2, u_1 \leqslant u_2$		(R7)
$\varphi\, \mathcal{U}_{[l_1,u_1]}\, \Box_{[0,u_2]}\varphi \mapsto \Box_{[l_1,l_1+u_2]}\varphi$	$\varphi\, \mathcal{U}_{[l_1,u_1]}\, \Diamond_{[0,u_2]}\varphi \mapsto \Diamond_{[l_1,l_1+u_2]}\varphi$	(R8)

Figure 2

Theorem 1. (Equivalence of MLTL Rewrite Rules) *Let $\varphi, \psi, \varphi_1, \varphi_2, \varphi_3$ be well-formed MLTL formulas and a, l, u, l_1, u_2, l_2, u_2, l_3, $u_3 \in \mathbb{N}_0$ such that $l \le u$, $l_1 \le u_1$, $l_2 \le u_2$, $l_3 \le u_3$. Then each rewrite relation ($\mapsto$) in Fig 2 is also an equivalence relation.*

Proof. We prove each rule in Figure 2 is semantics-preserving i.e., the right- and left-hand sides of the $\mapsto$ operator are equivalent. Recall that two MLTL formulas φ, ψ are equivalent only if $\pi \models \varphi \leftrightarrow \pi \models \psi$ for all π.

(R1) Let φ be an MLTL formula, π be a trace, and $l_1 \le u_1, l_2 \le u_2$. We prove

$$\pi \models \Box_{[l_1,u_1]}\Box_{[l_2,u_2]}\varphi \leftrightarrow \pi \models \Box_{[l_1+l_2,u_1+u_2]}\varphi.$$

($\rightarrow$) Let π be defined such that $\pi \models \Box_{[l_1,u_1]}\Box_{[l_2,u_2]}\varphi$. We show that $\pi \models \Box_{[l_1+l_2,u_1+u_2]}\varphi$ using the MLTL semantics. By the semantics of $\Box_I$, we know that for each $i \in [l_1,u_1]$, $\pi_i \models \Box_{[l_2,u_2]}\varphi$. Intuitively, this means that π satisfies φ at timestamps $[l_2,u_2]$ relative to i i.e., $\pi \models \varphi$ starting at timestamp $i + l_2$ and ending at timestamp $i + u_2$. So, applying the semantics of $\Box_I$ again, we have that $\pi_{i+j} \models \varphi$ for each $i \in [l_1,u_1]$, $j \in [l_2,u_2]$. By the definition of trace suffixes, this means that $\pi_k \models \varphi$ for each $k \in [l_1+l_2, u_1+u_2]$. Therefore $\pi \models \Box_{[l_1+l_2,u_1+u_2]}\varphi$.

($\leftarrow$) Let π be defined such that $\pi \models \Box_{[l_1+l_2,u_1+u_2]}\varphi$. We show that $\pi \models \Box_{[l_1,u_1]}\Box_{[l_2,u_2]}\varphi$. Then $\pi_k \models \varphi$ for all $k \in [l_1+l_2, u_1+u_2]$. Splitting this interval into two intervals, we have that $\pi_{i+j} \models \varphi$ for each $i \in [l_1,u_1]$, $j \in [l_2,u_2]$, as in the converse proof. Then $\pi \models \Box_{[l_1,u_1]}\Box_{[l_2,u_2]}\varphi$ by the semantics of $\Box_I$.

The proof for the $\Diamond$ version of (R1) is symmetric.

(R2): Let φ, ψ be MLTL formulas and $l_1 \leq u_1, l_2 \leq u_2$. We prove

$$\Box_{[l_1,u_1]}\varphi \vee \Box_{[l_2,u_2]}\psi \equiv \Box_{[l_3,u_3]}(\Box_{[l_1-l_3,u_1-u_3]}\varphi \vee \Box_{[l_2-l_3,u_2-u_3]}\psi)$$

for any $l_3 = min(l_1, l_2)$, $u_3 = l_3 + min(u_1 - l_1, u_2 - l_2)$, $l_3 < u_3$ using established equivalences and the MLTL semantics. Using (R1), we see that

$$\Box_{[l_1,u_1]}\varphi \wedge \Box_{[l_2,u_2]}\psi \equiv \Box_{[l_3,l_3]}\Box_{[l_1-l_3,u_1-l_3]}\varphi \wedge \Box_{[l_3,l_3]}\Box_{[l_2-l_3,u_2-l_3]}\psi.$$

This follows if both intervals $[l_1 - l_3, u_1 - l_3]$, $[l_2 - l_3, u_2 - l_3]$ are valid i.e., (a) $l_1 - l_3 \geq 0$, (b) $l_2 - l_3 \geq 0$, and (c) $l_1 - l_3 \leq u_1 - l_3$, (d) $l_2 - l_3 \leq u_2 - l_3$.

(a) Recall that $l_3 = l_1$, then $l_3 \leq l_1$.
(b) Recall that $l_3 = l_1 \leq l_2$, then $l_2 - l_3 \geq 0 \rightarrow l_2 \geq l_3 \rightarrow l_3 \leq l_2$ holds.
(c) Since $l_1 \leq u_1$, we see that $l_1 - l_3 \leq u_1 - l_3 \rightarrow l_1 \leq u_1$ holds.
(d) Since $l_2 \leq u_2$, we see that $l_2 - l_3 \leq u_2 - l_3 \rightarrow l_2 \leq u_2$ holds.

Now, let $u_3 = l_3 + min(u_1 - l_1, u_2 - l_2)$. Applying (R1) once more, we have

$$\Box_{[l_3,l_3]}\Box_{[l_1-l_3,u_1-l_3]}\varphi \wedge \Box_{[l_3,l_3]}\Box_{[l_2-l_3,u_2-l_3]}\psi \equiv \\ \Box_{[l_3,u_3]}\Box_{[l_1-l_3,u_1-u_3]}\varphi \wedge \Box_{[l_3,u_3]}\Box_{[l_2-l_3,u_2-u_3]}\psi$$

Since this only affects the upper bounds of the inner $\Box$ operators, we show that (a) $l_1 - l_3 \leq u_1 - u_3$ and (b) $l_2 - l_3 \leq u_2 - u_3$.

(a) Consider the two cases of $u_1 - l_1 \leq u_2 - l_2$ and $u_2 - l_2 < u_1 - l_1$:
   i. Assume $u_1 - l_1 \leq u_2 - l_2$, then $u_3 = u_1 - l_1 + l_3$. Replacing this in the target inequality, we have

   $$l_1 - l_3 \leq u_1 - (u_1 - l_1 + l_3) \rightarrow l_1 - l_3 \leq l_1 - l_3.$$

   ii. Otherwise, $u_2 - l_2 < u_1 - l_1$. Then $u_3 = u_2 - l_2 + l_3$, and replacing this in the target inequality, we have

   $$l_1 - l_3 \leq u_1 - (u_2 - l_2 + l_3) \rightarrow 0 \leq u_1 - l_3 - (u_2 - l_2) \rightarrow u_2 - l_2 \leq u_1 - l_3.$$

   Now, since $l_3 = l_1$, we have $u_2 - l_2 \leq u_1 - l_1$ which is true from our assumption.

(b) Consider the two cases of $u_1 - l_1 \leq u_2 - l_2$ and $u_2 - l_2 < u_1 - l_1$:
   i. Assume $u_1 - l_1 \leq u_2 - l_2$, then $u_3 = u_1 - l_1 + l_3$. Replacing this in the target inequality, we have

   $$l_2 - l_3 \leq u_2 - (u_1 - l_1 + l_3) \rightarrow l_2 - l_3 \leq u_2 - u_1 + l_1 - l_3 \rightarrow \\ l_2 \leq u_2 - u_1 + l_1 \rightarrow u_1 - l_1 \leq u_2 - l_2,$$

   which is true from our assumption.

ii. Otherwise, $u_2 - l_2 < u_1 - l_1$, then $u_3 = u_2 - l_2 + l_3$. Replacing this in the target inequality, we have

$$l_2 - l_3 \leq u_2 - (u_2 - l_2 + l_3) \rightarrow l_2 - l_3 \leq u_2 - u_2 + l_2 - l_3 \rightarrow$$
$$l_2 \leq u_2 - u_2 + l_2 \rightarrow l_2 \leq l_2.$$

Finally, let π be a trace. We prove that

$$\pi \models \Box_{[l_3,u_3]}\Box_{[l_1-l_3,u_1-l_3-u_3]}\varphi \wedge \Box_{[l_3,u_3]}\Box_{[l_2-l_3,u_2-l_3-u_3]}\psi \leftrightarrow$$
$$\pi \models \Box_{[l_3,u_3]}(\Box_{[l_1-l_3,u_1-l_3-u_3]}\varphi \wedge \Box_{[l_2-l_3,u_2-l_3-u_3]}\psi).$$

(a) ($\rightarrow$) Let π be defined such that $\pi \models (\Box_{[l_3,u_3]}\Box_{[l_1,u_1]}\varphi) \wedge (\Box_{[l_3,u_3]}\Box_{[l_2,u_2]}\psi)$. We show that $\pi \models \Box_{[l_3,u_3]}(\Box_{[l_1-l_3,u_1-l_3-u_3]}\varphi \wedge \Box_{[l_2-l_3,u_2-l_3-u_3]}\psi)$ using the MLTL semantics. We apply the semantic definitions of $\wedge$ and $\Box_I$ to see that $\pi_i \models \Box_{[l_1,u_1]}\varphi$ and $\pi_i \models \Box_{[l_2,u_2]}\psi$ for all $i \in [l_3, u_3]$. Combining these relations using the semantics of $\wedge$ once more, we see that $\pi_i \models \Box_{[l_1,u_1]}\varphi \wedge \Box_{[l_2,u_2]}\psi$ for all $i \in [l_3, u_3]$. Using the semantics of $\Box_I$ again, we see that $\pi \models \Box_{[l_3,u_3]}(\Box_{[l_1,u_1]}\varphi \wedge \Box_{[l_2,u_2]}\psi)$.

(b) ($\leftarrow$) Conversely, let π be defined such that $\pi \models \Box_{[l_3,u_3]}(\Box_{[l_1,u_1]}\varphi \wedge \Box_{[l_2,u_2]}\psi)$. We show that $\pi \models (\Box_{[l_3,u_3]}\Box_{[l_1,u_1]}\varphi) \wedge (\Box_{[l_3,u_3]}\Box_{[l_2,u_2]}\psi)$ using the MLTL semantics. Then $\pi_i \models \Box_{[l_1,u_1]}\varphi$ and $\pi_i \models \Box_{[l_2,u_2]}\psi$ for all $i \in [l_3, u_3]$. Using the semantic definitions of $\wedge$ and $\Box_I$, we see that $\pi \models \Box_{[l_3,u_3]}\Box_{[l_1,u_1]}\varphi$ and $\pi \models \Box_{[l_3,u_3]}\Box_{[l_2,u_2]}\psi$, so $\pi \models (\Box_{[l_3,u_3]}\Box_{[l_1,u_1]}\varphi) \wedge (\Box_{[l_3,u_3]}\Box_{[l_2,u_2]}\psi)$.

The proof for the $\Diamond$ version of (R2) is symmetric.

(R3): Let φ be a MLTL formula and $l \leq u$. We prove

$$\Box_{[a,a]}\Diamond_{[l,u]}\varphi \equiv \Diamond_{[l,u]}\Box_{[a,a]}\varphi$$

using established MLTL equivalences. Using Equation 3 and (R1) to expand the expression until we have a of the $\Box_{[1,1]}$ operators in lines 3 and 9 of the following proof we can show:

$$\begin{aligned}
\Box_{[a,a]}\Diamond_{[l,u]}\varphi &\equiv \Box_{[1,1]}\Box_{[a-1,a-1]}\Diamond_{[l,u]}\varphi \\
&\equiv \cdots \text{Applying } a \text{ times} \\
&\equiv \Box_{[1,1]}\cdots\Box_{[1,1]}\Diamond_{[l,u]}\varphi \\
&\equiv \Diamond_{[1,1]}\cdots\Diamond_{[1,1]}\Diamond_{[l,u]}\varphi \\
&\equiv \Diamond_{[l+a,u+a]}\varphi \\
&\equiv \Diamond_{[l+a-1,u+a-1]}\Diamond_{[1,1]}\varphi \\
&\equiv \cdots \text{Applying } a \text{ times} \\
&\equiv \Diamond_{[l,u]}\Diamond_{[1,1]}\cdots\Diamond_{[1,1]}\varphi \\
&\equiv \Diamond_{[l,u]}\Box_{[1,1]}\cdots\Box_{[1,1]}\varphi \\
&\equiv \Diamond_{[l,u]}\Box_{[a,a]}\varphi.
\end{aligned}$$

The proof for the $\Diamond_{[a,a]}$ version of (R3) is symmetric.

(R4): Let φ be a MLTL formula and $l_1 \leq u_1$, $l_2 \leq u_2$, $l_1 \leq l_2 \leq u_1 + 1$, $u_3 = max(u_1, u_2)$. We prove that

$$\Box_{[l_1,u_1]}\varphi \wedge \Box_{[l_2,u_2]}\varphi \equiv \Box_{[l_1,u_3]}\varphi.$$

We first show that $\pi \models \Box_{[l,l]}\varphi \wedge \Box_{[l+1,u]}\varphi \leftrightarrow \pi \models \Box_{[l,u]}\varphi$ for any trace pi and $l < u$.

(a) ($\rightarrow$) Let π be a trace such that $\pi \models \Box_{[l,l]}\varphi \wedge \Box_{[l+1,u]}\varphi$. We show that $\pi \models \Box_{[l,u]}\varphi$. From the semantics of $\Box$ and (R1), we see that $\pi_i \models \varphi$ for all $i \in [l,l]$ and $\pi_j \models \varphi$ for all $j \in [l+1,u]$. Now, since $[l,l] \cup [l+1,u] = [l,u]$, it follows that $\pi_k \models \varphi$ for all $k \in [l,u]$. This matches the semantic definition of $\Box$ and therefore $\pi \models \Box_{[l,u]}\varphi$.

(b) ($\leftarrow$) Conversely, if π is a trace such that $\pi \models \Box_{[l,u]}\varphi$, then $\pi \models \Box_{[l,l]}\varphi \wedge \Box_{[l+1,u]}\varphi$ because $\pi_i \models \varphi$ for all $i \in [l,u]$ where $[l,l] \cup [l+1,u] = [l,u]$ as before.

From above, we can expand each $\Box_I$ operator to a conjunction of singleton intervals, remove repeated conjunctive clauses, then use the above equivalence again to simplify:

$$\begin{aligned}
\Box_{[l_1,u_1]}\varphi \wedge \Box_{[l_2,u_2]}\varphi \equiv & (\Box_{[l_1,l_1]}\varphi \wedge \Box_{[l_1+1,u_1]}) \wedge \Box_{[l_2,u_2]}\varphi \\
\equiv & \cdots \text{Applying } u_1 - l_1 \text{ times} \\
\equiv & (\Box_{[l_1,l_1]}\varphi \wedge \cdots \wedge \Box_{[u_1,u_1]}\varphi) \wedge \Box_{[l_2,u_2]}\varphi \\
\equiv & (\Box_{[l_1,l_1]}\varphi \wedge \cdots \wedge \Box_{[u_1,u_1]}\varphi) \wedge (\Box_{[l_2,l_2]}\varphi \wedge \Box_{[l_2+1,u_2]}\varphi) \\
\equiv & \cdots \text{Applying } u_2 - l_2 \text{ times} \\
\equiv & \Box_{[l_1,l_1]}\varphi \wedge \cdots \wedge \Box_{[u_1,u_1]}\varphi \wedge (\Box_{[l_2,l_2]}\varphi \wedge \cdots \wedge \Box_{[u_2,u_2]}\varphi) \\
\equiv & \Box_{[l_1,l_1]}\varphi \wedge \cdots \wedge \Box_{[l_2-1,l_2-1]}\varphi \wedge \Box_{[l_2,l_2]}\varphi \wedge \cdots \wedge \Box_{[u_2,u_2]}\varphi \\
\equiv & \Box_{[l_1,l_1+1]}\varphi \wedge \cdots \wedge \Box_{[l_2-1,l_2-1]}\varphi \wedge \Box_{[l_2,l_2]}\varphi \wedge \cdots \wedge \Box_{[u_2,u_2]}\varphi \\
\equiv & \cdots \text{Applying } l_2 - l_1 - 1 \text{ times} \\
\equiv & \Box_{[l_1,l_2-1]}\varphi \wedge \Box_{[l_2,l_2]}\varphi \wedge \cdots \wedge \Box_{[u_2,u_2]}\varphi \\
\equiv & \cdots \text{Applying } u_2 - l_2 + 1 \text{ times} \\
\equiv & \Box_{[l_1,u_2]}\varphi
\end{aligned}$$

Note that from lines 6 to 7 we remove repeated clauses e.g., $l_2 \leq u_1 \leq u_2$ so there must be two instances of the expression $\Box_{[u_1,u_1]}\varphi$ in the formula.

The proof for the $\Diamond$ version of (R4) is symmetric.

(R5): Let φ be a MLTL formula and $l_1 \leq l_2 \leq u_2 \leq u_1$. We prove that

$$\Box_{[l_1,u_1]}\varphi \vee \Box_{[l_2,u_2]}\varphi \equiv \Box_{[l_2,u_2]}\varphi.$$

We first show that

$$\pi \models \Box_{[l,u_1]}\varphi \vee \Box_{[l,u_2]}\varphi \leftrightarrow \pi \models \Box_{[l,u_1]}\varphi$$

for any trace π.

(a) (→) Assume for the purposes of contraction that π is a trace such that $\pi \models \Box_{[l,u_1]}\varphi \vee \Box_{[l,u_2]}\varphi$ but $\pi \not\models \Box_{[l,u_1]}\varphi$. Therefore there is some $i \in [l, u_1]$ such that $\pi_i \not\models \varphi$. By (R1), we see that

$$\Box_{[l,u_1]}\varphi \vee \Box_{[l,u_2]}\varphi \equiv \Box_{[l,u_1]}\varphi \vee \Box_{[0,u_2-u_1]}\Box_{[l,u_1]}\varphi$$

But $\pi \not\models \Box_{[l,u_1]}\varphi \vee \Box_{0,u_2-u_1}\Box_{[l,u_1]}\varphi$ since $\pi \not\models \Box_{[l,u_1]}\varphi$. Therefore if $\pi \models \Box_{[l,u_1]}\varphi \vee \Box_{[l,u_2]}\varphi$, then $\pi \models \Box_{[l,u_1]}\varphi$.

(b) (←) Conversely, assume that π is a trace such that $\pi \models \Box_{[l,u_1]}\varphi$ but $\pi \not\models \Box_{[l,u_1]}\varphi \vee \Box_{[l,u_2]}\varphi$. But this is a contradiction, since by the definition of disjunction, if $\pi \models \Box_{[l,u_1]}\varphi$, then $\pi \models \Box_{[l,u_1]}\varphi \vee \Box_{[l,u_2]}\varphi$ since the left-hand disjunctive clause models π. Therefore if $\pi \models \Box_{[l,u_1]}\varphi$, then $\pi \models \Box_{[l,u_1]}\varphi \vee \Box_{[l,u_2]}\varphi$.

Next we consider the cases for when $l_1 < l_2$ and $l_1 = l_2$.

(a) ($l_1 < l_2$) Starting from the left-hand side of the equivalence, we use (R3), the above equivalence, and the Absorption Law of Propositional Logic to show that:

$$\begin{aligned}\Box_{[l_1,u_1]}\varphi \vee \Box_{[l_2,u_2]}\varphi &\equiv (\Box_{[l_1,l_2-1]}\varphi \wedge \Box_{[l_2,u_1]}\varphi) \vee \Box_{[l_2,u_2]}\varphi \\ &\equiv (\Box_{[l_1,l_2-1]}\varphi \wedge \Box_{[l_2,u_1]}\varphi) \vee \Box_{[l_2,u_2]}\varphi \\ &\equiv (\Box_{[l_1,l_2-1]}\varphi \vee \Box_{[l_2,u_2]}\varphi) \wedge (\Box_{[l_2,u_1]}\varphi \vee \Box_{[l_2,u_2]}\varphi) \\ &\equiv (\Box_{[l_1,l_2-1]}\varphi \vee \Box_{[l_2,u_2]}\varphi) \wedge \Box_{[l_2,u_2]}\varphi \\ &\equiv \Box_{[l_2,u_2]}\varphi\end{aligned}$$

(b) ($l_1 = l_2$) Then starting with the left-hand side, replacing l_2 with l_1 and using the above equivalence we show:

$$\begin{aligned}\Box_{[l_1,u_1]}\varphi \vee \Box_{[l_2,u_2]}\varphi &\equiv \Box_{[l_1,u_1]}\varphi \vee \Box_{[l_1,u_2]}\varphi \\ &\equiv \Box_{[l_2,u_2]}\varphi\end{aligned}$$

(R6): Let φ, ψ be MLTL formulas, π be a trace, and $l \leq u$. We prove that

$$\pi \models \Box_{[a,a]}(\varphi\, \mathcal{U}_{[l,u]}\psi) \leftrightarrow \pi \models \varphi\, \mathcal{U}_{[l+a,u+a]}\psi$$

(a) (→) Let π be a trace such that $\pi \models \Box_{[a,a]}(\varphi\, \mathcal{U}_{[l,u]}\psi)$. We show that $\pi \models \varphi\, \mathcal{U}_{[l+a,u+a]}\psi$. By the semantics of $\Box_{[a,a]}$, we see that $\pi_a \models \varphi\, \mathcal{U}_{[l,u]}\psi$. Similar to the proof for (R1), we say that $\pi_a \models \varphi\, \mathcal{U}_{[l,u]}\psi$ relative to the bounds of $[l, u]$. For instance, if $\pi_{a+l} \models \psi$, then $\pi_a \models \varphi\, \mathcal{U}_{[l,u]}\psi$. We then say that there is some $i \in [l, u]$ such that $\pi_{a+i} \models \psi$ and for all $j \in [l, u]$ such that $j < i$, $\pi_{a+j} \models \varphi$. This directly corresponds to the semantics of $\mathcal{U}_{[l+a,u+a]}$, so therefore $\pi \models \varphi\, \mathcal{U}_{[l+a,u+a]}\psi$.

(b) (←) Conversely, let π be a trace such that $\pi \models \varphi\, \mathcal{U}_{[l+a,u+a]}\psi$. Then $\pi_i \models \psi$ for some $i \in [l + a, u + a]$ and $\pi_j \models \varphi$ for all $j \in [l + a, u + a]$ such that $j < i$. Therefore it follows that $\pi \models \Box_{[a,a]}(\varphi\, \mathcal{U}_{[l,u]}\psi)$ from the MLTL semantics.

The proof for the $\Diamond_{[a,a]}$ version of (R6) is symmetric.

(R7): Let $\varphi_1, \varphi_2, \varphi_3$ be MLTL formulas, π be a trace, and $l \leq u_1$, $l \leq u_2$, $u_1 \leq u_2$. We prove that

$$\pi \models (\varphi_1 \; \mathcal{U}_{[l,u_1]} \varphi_2) \wedge (\varphi_3 \; \mathcal{U}_{[l,u_2]} \varphi_2) \leftrightarrow \pi \models (\varphi_1 \wedge \varphi_3) \; \mathcal{U}_{[l,u_1]} \varphi_2.$$

(a) ($\rightarrow$) Let π be such that $\pi \models (\varphi_1 \; \mathcal{U}_{[l,u_1]} \varphi_2) \wedge (\varphi_3 \; \mathcal{U}_{[l,u_2]} \varphi_2)$. We show that $\pi \models (\varphi_1 \wedge \varphi_3) \; \mathcal{U}_{[l,u_1]} \varphi_2$. From the semantics of $\mathcal{U}_I$, there must be some $i \in [l, u_1]$ such that $\pi_i \models \varphi_2$ in order to satisfy the clause $\varphi_1 \; \mathcal{U}_{[l,u_1]} \varphi_2$. Further, using the relation $u_1 \leq u_2$, we know that $\pi_j \models \varphi_1$ and $\pi_j \models \varphi_3$ for all $j \in [l, u_1] \subseteq [l, u_2]$ such that $j < i$. Putting this all together using the semantic definition of $\mathcal{U}_I$, $\pi \models (\varphi_1 \wedge \varphi_3) \; \mathcal{U}_{[l,u_1]} \varphi_2$.

(b) ($\leftarrow$) Conversely, let π be such that $\pi \models (\varphi_1 \wedge \varphi_3) \; \mathcal{U}_{[l,u_1]} \varphi_2$. Then $\pi \models (\varphi_1 \; \mathcal{U}_{[l,u_1]} \varphi_2) \wedge (\varphi_3 \; \mathcal{U}_{[l,u_2]} \varphi_2)$ because $\pi_i \models \varphi_2$ for some $i \in [l, u_1] \subseteq [l, u_2]$ and $\pi_j \models \varphi_1$ and $\pi_j \models \varphi_3$ for all $j \in [l, u_1]$ such that $j < i$.

(R8): Let φ be a MLTL formula, and $l_1 \leq u_1$. We prove that

$$\varphi \; \mathcal{U}_{[l_1,u_1]} \Box_{[0,u_2]} \varphi \equiv \Box_{[l_1, l_1+u_2]} \varphi.$$

We first show that

$$\Box_{[l,l]} \psi \vee (\Box_{[l,l]} \varphi \wedge (\varphi \; \mathcal{U}_{[l+1,u]} \psi)) \equiv \varphi \; \mathcal{U}_{[l,u]} \psi$$

for any trace π.

(a) Let π be such that $\pi \models \Box_{[l,l]} \psi \vee (\Box_{[l,l]} \varphi \wedge (\varphi \; \mathcal{U}_{[l+1,u]} \psi))$. We show that $\pi \models \varphi \; \mathcal{U}_{[l,u]} \psi$. Distributing the $\Box_{[l,l]} \psi$ in the left-hand expression, we obtain the formula $(\Box_{[l,l]} \psi \vee \Box_{[l,l]} \varphi) \wedge (\Box_{[l,l]} \psi \vee \varphi \; \mathcal{U}_{[l+1,u]} \psi)$. Consider the case where $\pi_l \models \psi$, then $\pi \models \varphi \; \mathcal{U}_{[l,u]} \psi$ by the semantics of $\mathcal{U}_I$. Otherwise, $\pi_l \not\models \psi$, then $\pi_l \models \varphi$ and $\varphi \; \mathcal{U}_{[l+1,u]} \psi$. It follows that $\pi \models \varphi \; \mathcal{U}_{[l,u]} \psi$ by the semantics of $\mathcal{U}_I$.

(b) Conversely, let π be a trace such that $\pi \not\models \Box_{[l,l]} \psi \vee (\Box_{[l,l]} \varphi \wedge (\varphi \; \mathcal{U}_{[l+1,u]} \psi))$. Then $\pi \not\models \varphi \; \mathcal{U}_{[l,u]} \psi$ because either $\pi_l \not\models \psi$ and $\pi_l \not\models \varphi$ or $\pi \not\models \varphi \; \mathcal{U}_{[l+1,u]} \psi$.

Now using the above equivalence, (R4), (R5), and Equation 3:

$$
\begin{aligned}
\varphi\,\mathcal{U}_{[l,u_1]}(\Box_{[0,u_2]}\varphi) \equiv & \Box_{[l,l]}\Box_{[0,u_2]}\varphi \vee (\Box_{[l,l]}\varphi \wedge (\varphi\,\mathcal{U}_{[l+1,u_1]}\Box_{[0,u_2]}\varphi)) \\
\equiv & \Box_{[l,l+u_2]}\varphi \vee (\Box_{[l,l]}\varphi \wedge (\varphi\,\mathcal{U}_{[l+1,u_1]}\Box_{[0,u_2]}\varphi)) \\
\equiv & \Box_{[l,l+u_2]}\varphi \vee (\Box_{[l,l]}\varphi \wedge (\Box_{[l+1,l+1+u_2]}\varphi \\
& \quad \vee (\Box_{[l+1,l+1]}\varphi \wedge (\varphi\,\mathcal{U}_{[l+2,u_1]}\Box_{[0,u_2]}\varphi))) \\
\equiv & \cdots \text{(Applying } u_1 \text{ times)} \\
\equiv & \Box_{[l,l+u_2]}\varphi \vee (\Box_{[l,l]}\varphi \wedge (\cdots \wedge (\Box_{[u_1-1,u_1+u_2-1]}\varphi \vee \\
& \quad (\Box_{[u_1-1,u_1-1]}\varphi \wedge \varphi\,\mathcal{U}_{[u_1,u_1]}\Box_{[0,u_2]}\varphi)))) \\
\equiv & \Box_{[l,l+u_2]}\varphi \vee (\Box_{[l,l]}\varphi \wedge (\cdots \wedge (\Box_{[u_1-1,u_1+u_2-1]}\varphi \vee \\
& \quad (\Box_{[u_1-1,u_1-1]}\varphi \wedge \Box_{[u_1,u_1]}\Box_{[0,u_2]}\varphi)))) \\
\equiv & \Box_{[l,l+u_2]}\varphi \vee (\Box_{[l,l]}\varphi \wedge (\cdots \wedge (\Box_{[u_1-1,u_1+u_2-1]}\varphi \vee \\
& \quad (\Box_{[u_1-1,u_1-1]}\varphi \wedge \Box_{[u_1,u_1+u_2]}\varphi)))) \\
\equiv & \Box_{[l,l+u_2]}\varphi \vee (\Box_{[l,l]}\varphi \wedge (\cdots \wedge (\Box_{[u_1-1,u_1+u_2-1]}\varphi \vee \\
& \quad \Box_{[u_1-1,u_1+u_2]}\varphi))) \\
\equiv & \Box_{[l,l+u_2]}\varphi \vee (\Box_{[l,l]}\varphi \wedge (\cdots \wedge \Box_{[u_1-1,u_1+u_2]}\varphi)) \\
\equiv & \cdots \text{(Applying } u_1 \text{ times)} \\
\equiv & \Box_{[l,l+u_2]}\varphi \vee (\Box_{[l,u_1+u_2]}\varphi) \\
\equiv & \Box_{[l,l+u_2]}\varphi
\end{aligned}
$$

□

Theorem 2. (Memory Reduction of Rewriting Rules) *Let φ, ψ_1, ψ_2 be well-formed MLTL formulas where ψ_1 is a sub-formula of φ. Then applying a valid rewrite rule in Figure 1 to ψ_1 will result in a new formula $\varphi(\psi_1 \mapsto \psi_2)$ such that $\varphi \equiv \varphi(\psi_1 \mapsto \psi_2)$ and*

$$mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) \leq mem_{AST}(\varphi)$$

Proof. (R1): Then $\psi_1 = \Box_{[l_1,u_1]}\Box_{[l_2,u_2]}\varphi$ and $\psi_2 = \Box_{[l_1+l_2,u_1+u_2]}\varphi$ where $l_1 \leq u_1$ and $l_2 \leq u_2$. Therefore, because $\psi_1.wpd = \varphi.wpd + u_1 + u_2 = \psi_2.wpd$ and $\psi_1 \equiv \psi_2$ by Theorem 1, we have $mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2) \leq mem_{AST}(\varphi) - mem_{AST}(\psi_1)$.

Next, using Lemma 1 we see that $mem_{node}(\Box_{[l_1,u_1]}) = mem_{node}(\Box_{[l_1+l_2,u_1+u_2]})$

since $\psi_1.bpd = \varphi.bpd + l_1 + l_2 = \psi_2.bpd$. Then

$$
\begin{aligned}
mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) =&(mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2)) + mem_{AST}(\psi_2)\\
=&(mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2))+\\
&\quad mem_{node}(\Box_{[l_1+l_2,u_1+u_2]}) + mem_{AST}(\varphi)\\
\leq&(mem_{AST}(\varphi) - mem_{AST}(\psi_1))+\\
&\quad mem_{node}(\Box_{[l_1+l_2,u_1+u_2]}) + mem_{AST}(\varphi)\\
=&(mem_{AST}(\varphi) - mem_{AST}(\psi_1))+\\
&\quad mem_{node}(\Box_{[l_1,u_1]}) + mem_{AST}(\varphi)\\
\leq&(mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + mem_{node}(\Box_{[l_1,u_1]})+\\
&\quad mem_{node}(\Box_{[l_2,u_2]}) + mem_{AST}(\varphi)\\
=&(mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + mem_{AST}(\psi_1)\\
=&mem_{AST}(\varphi)
\end{aligned}
$$

The proof for the rule $\Diamond_{[l_1,u_1]}\Diamond_{[l_2,u_2]}\varphi \to \Diamond_{[l_1+l_2,u_1+u_2]}\varphi$ follows.

(R2): Then $\psi_1 = \Box_{[l_1,u_1]}\varphi_1 \wedge \Box_{[l_2,u_2]}\varphi_2$ and $\psi_2 = \Box_{[l_3,u_3]}(\Box_{[l_1-l_3,u_1-u_3]}\varphi_1 \wedge \Box_{[l_2-l_3,u_2-u_3]}\varphi_2)$ where $l_1 \leq u_1$, $l_2 \leq u_2$, $l_3 = min(l_1, l_2)$, $u_3 = l_3 + min(u_1 - l_1, u_2 - l_2)$. We show that

$$
\begin{aligned}
\psi_1.wpd =&max(\varphi.wpd + u_1, \varphi.wpd + u_2)\\
=&max(\varphi.wpd + u_1 + (u_3 - u_3), \varphi.wpd + u_2 + (u_3 - u_3))\\
=&max(\varphi.wpd + u_3 + (u_1 - u_3), \varphi.wpd + u_3 + (u_2 - u_3))\\
=&u_3 + max(\varphi.wpd + (u_1 - u_3), \varphi.wpd + (u_2 - u_3))\\
=&\psi_2.wpd
\end{aligned}
$$

Therefore we have $\psi_1 \equiv \psi_2$ by Theorem 1 so that $mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2) \leq mem_{AST}(\varphi) - mem_{AST}(\psi_1)$ by Lemma 2.

Next we show that

$$
\begin{aligned}
\psi_1.bpd =&min(\varphi.bpd + l_1, \varphi.wpd + l_2)\\
=&min(\varphi.bpd + l_1 + (l_3 - l_3), \varphi.wpd + l_2 + (l_3 - l_3))\\
=&min(\varphi.bpd + l_3 + (l_1 - l_3), \varphi.wpd + l_3 + (l_2 - l_3))\\
=&l_3 + min(\varphi.bpd + (l_1 - l_3), \varphi.wpd + (l_2 - l_3))\\
=&\psi_2.bpd
\end{aligned}
$$

Using Lemma 1 we see that $mem_{node}(\wedge_{\psi_1}) = mem_{node}(\Box_{[l_3,u_3]})$ since $\psi_1.bpd = \psi_2.bpd$. Then

$$
\begin{aligned}
mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) =&(mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2)) + mem_{AST}(\psi_2)\\
=&(mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2)) + mem_{node}(\wedge_{\psi_2})\\
&mem_{node}(\Box_{[l_3,u_3]}) + mem_{node}(\Box_{[l_1-l_3,u_1-u_3]})+\\
&mem_{AST}(\varphi_1) + mem_{node}(\Box_{[l_2-l_3,u_2-u_3]}) + mem_{AST}(\varphi_2)\\
\leq&(mem_{AST}(\varphi) - mem_{AST}(\psi_1))+
\end{aligned}
$$

The proof follows for $\Diamond_{[l_1,u_1]}\varphi \vee \Diamond_{[l_2,u_2]}\psi \to \Diamond_{[l_3,u_3]}(\Diamond_{[l_1-l_3,u_1-u_3]}\varphi \vee \Diamond_{[l_2-l_3,u_2-u_3]}\psi)$.

(R3): Then $\psi_1 = \Box_{[a,a]}\Diamond_{[l,u]}\varphi$ and $\psi_2 = \Diamond_{[l+a,u+a]}\varphi$ where $l \leq u$. Therefore, because $\psi_1.wpd = \varphi.wpd + u + a = \psi_2.wpd$ and $\psi_1 \equiv \psi_2$ by Theorem 1, we have $mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2) \leq mem_{AST}(\varphi) - mem_{AST}(\psi_1)$ by Lemma 2.

Next, using Lemma 1 we see that $mem_{node}(\Box_{[a,a]}) = mem_{node}(\Diamond_{[l+a,u+a]})$ since $\psi_1.bpd = \varphi.bpd + l + a = \psi_2.bpd$. Then

$$\begin{aligned}
mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) =& (mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2)) + mem_{AST}(\psi_2) \\
=& (mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2)) + \\
& \quad mem_{node}(\Diamond_{[l+a,u+a]}) + mem_{AST}(\varphi) \\
\leq& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + \\
& \quad mem_{node}(\Diamond_{[l+a,u+a]}) + mem_{AST}(\varphi) \\
=& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + \\
& \quad mem_{node}(\Box_{[a,a]}) + mem_{AST}(\varphi) \\
\leq& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + mem_{node}(\Box_{[a,a]}) + \\
& \quad mem_{node}(\Diamond_{[l,u]}) + mem_{AST}(\varphi) \\
=& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + mem_{AST}(\psi_1) \\
=& mem_{AST}(\varphi)
\end{aligned}$$

The proof follows for $\Diamond_{[l,u]}\Box_{[a,a]}\varphi \to \Diamond_{[l+a,u+a]}\varphi$, $\Diamond_{[a,a]}\Box_{[l,u]}\varphi \to \Box_{[l+a,u+a]}\varphi$, $\Box_{[l,u]}\Diamond_{[a,a]}\varphi \to \Box_{[l+a,u+a]}\varphi$ follows.

(R4): Then $\psi_1 = \Box_{[l_1,u_1]}\varphi \wedge \Box_{[l_2,u_2]}\varphi$ and $\psi_2 = \Box_{[l_1,u_3]}\varphi$ where $l_1 \leq u_1, l_2 \leq u_2$, $l_1 \leq l_2 \leq u_1 + 1$, and $u_3 = max(u_1, u_2)$. We show that

$$\begin{aligned}
\psi_1.wpd =& max(\varphi.wpd + u_1, \varphi.wpd + u_2) \\
=& \varphi.wpd + max(u_1, u_2) \\
=& \varphi.wpd + u_3 \\
=& \psi_2.wpd
\end{aligned}$$

Therefore we have $\psi_1 \equiv \psi_2$ by Theorem 1 so that $mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2) \leq mem_{AST}(\varphi) - mem_{AST}(\psi_1)$ by Lemma 2.

Next we show that

$$\begin{aligned}
\psi_1.bpd =& min(\varphi.bpd + l_1, \varphi.wpd + l_2) \\
=& \varphi.bpd + min(l_1, l_2) \\
=& \varphi.bpd + l_1 \\
=& \psi_2.bpd
\end{aligned}$$

Using Lemma 1 we see that $mem_{node}(\wedge) = mem_{node}(\Box_{[l_1,u_3]})$ since $\psi_1.bpd = \psi_2.bpd$. Then

$$\begin{aligned}
mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) =&(mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2)) + mem_{AST}(\psi_2) \\
=&(mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2))+ \\
&\quad mem_{node}(\Box_{[l_1,u_3]}) + mem_{AST}(\varphi) \\
\leq&(mem_{AST}(\varphi) - mem_{AST}(\psi_1))+ \\
&\quad mem_{node}(\Box_{[l_3,u_3]}) + mem_{AST}(\varphi) \\
=&(mem_{AST}(\varphi) - mem_{AST}(\psi_1))+ \\
&\quad mem_{node}(\wedge) + mem_{AST}(\varphi) \\
\leq&(mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + mem_{node}(\wedge)+ \\
&\quad mem_{node}(\Box_{[l_1,u_1]}) + mem_{node}(\Box_{[l_2,u_2]}) + 2 \cdot mem_{AST}(\varphi) \\
=&(mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + mem_{AST}(\psi_1) \\
=&mem_{AST}(\varphi)
\end{aligned}$$

The proof follows for $\Diamond_{[l_1,u_1]}\varphi \vee \Diamond_{[l_2,u_2]}\varphi \to \Diamond_{[l_1,u_3]}\varphi$.

(R5): Let $\psi_1 = \Box_{[l_1,u_1]}\varphi \vee \Box_{[l_2,u_2]}\varphi$ and $\psi_2 = \Box_{[l_3,u_3]}\varphi$ where $l_1 \leq l_2 \leq u_2 \leq u_1$ and $l_2 = l_3$, $u_2 = u_3$. We show that

$$\begin{aligned}
\psi_1.wpd =&max(\varphi.wpd + u_1, \varphi.wpd + u_2) \\
=&\varphi.wpd + max(u_1, u_2) \\
=&\varphi.wpd + u_1 \\
\geq&\varphi.wpd + u_2 \\
=&\varphi.wpd + u_3 \\
=&\psi_2.wpd
\end{aligned}$$

Therefore we have $\psi_1 \equiv \psi_2$ by Theorem 1 so that $mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2) \leq mem_{AST}(\varphi) - mem_{AST}(\psi_1)$ by Lemma 2.

Next we show that

$$\begin{aligned}
\psi_1.bpd =&min(\varphi.bpd + l_1, \varphi.wpd + l_2) \\
=&\varphi.bpd + min(l_1, l_2) \\
=&\varphi.bpd + l_1 \\
\leq&\varphi.bpd + l_2 \\
=&\varphi.bpd + l_3 \\
=&\psi_2.bpd
\end{aligned}$$

Using Lemma 1 we see that $mem_{node}(\vee) = mem_{node}(\Box_{[l_3,u_3]})$ since $\psi_1.bpd \leq$

$\psi_2.bpd$. Then

$$\begin{aligned}
mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) =& (mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2)) + mem_{AST}(\psi_2) \\
=& (mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2)) + \\
& \quad mem_{node}(\Box_{[l_3,u_3],\psi_2}) + mem_{AST}(\varphi) \\
\leq& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + \\
& \quad mem_{node}(\Box_{[l_3,u_3],\psi_2}) + mem_{AST}(\varphi) \\
=& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + \\
& \quad mem_{node}(\vee) + mem_{AST}(\varphi) \\
\leq& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + mem_{node}(\vee) + \\
& \quad mem_{node}(\Box_{[l_1,u_1]}) + mem_{node}(\Box_{[l_2,u_2]}) + 2 \cdot mem_{AST}(\varphi) \\
=& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + mem_{AST}(\psi_1) \\
=& mem_{AST}(\varphi)
\end{aligned}$$

(R6): Let $\psi_1 = \Box_{[a,a]}(\varphi \mathcal{U}_{[l,u]} \psi)$ and $\psi_2 = \varphi \mathcal{U}_{[l+a,u+a]} \psi$ where $l \leq u$. Therefore, because $\psi_1.wpd = max(\varphi.wpd, \psi.wpd) + u + a = max(\varphi.wpd + u + a, \psi.wpd + u + a) = \psi_2.wpd$ and $\psi_1 \equiv \psi_2$ by Theorem 1, we have $mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2) \leq mem_{AST}(\varphi) - mem_{AST}(\psi_1)$ by Lemma 2.

Next, using Lemma 1 we see that $mem_{node}(\Box_{[a,a]}) = mem_{node}(\mathcal{U}_{[l+a,u+a]})$ since $\psi_1.bpd = min(\varphi.bpd, \psi.bpd) + l + a = min(\varphi.bpd + l + a, \psi.bpd + l + a) = \psi_2.bpd$. Then

$$\begin{aligned}
mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) =& (mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2)) + mem_{AST}(\psi_2) \\
=& (mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2)) + \\
& \quad mem_{node}(\mathcal{U}_{[l+a,u+a]}) + mem_{AST}(\varphi) + mem_{AST}(\psi) \\
\leq& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + \\
& \quad mem_{node}(\mathcal{U}_{[l+a,u+a]}) + mem_{AST}(\varphi) + mem_{AST}(\psi) \\
=& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + \\
& \quad mem_{node}(\Box_{[a,a]}) + mem_{AST}(\varphi) + mem_{AST}(\psi) \\
\leq& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + mem_{node}(\Box_{[a,a]}) + \\
& \quad mem_{node}(\mathcal{U}_{[l,u]}) + mem_{AST}(\varphi) + mem_{AST}(\psi) \\
=& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + mem_{AST}(\psi_1) \\
=& mem_{AST}(\varphi)
\end{aligned}$$

The proof follows for $(\Box_{[a,a]}\varphi)\mathcal{U}_{[l,u]}(\Box_{[a,a]}\psi) \to \varphi \mathcal{U}_{[l+a,u+a]} \psi$.

(R7): Let $\psi_1 = (\varphi_1 \mathcal{U}_{[l,u_1]} \varphi_2) \wedge (\varphi_3 \mathcal{U}_{[l,u_2]} \varphi_2)$ and $\psi_2 = (\varphi_1 \wedge \varphi_3) \mathcal{U}_{[l,u_3]} \varphi_2$ where

$l \leq u_1$, $l \leq u_2$, $u_1 \leq u_2$, and $u_3 = u_1$. We show that

$$\begin{aligned}
\psi_1.wpd =& max(max(\varphi_1.wpd, \varphi_2.wpd) + u_1, max(\varphi_3.wpd, \varphi_2.wpd) + u_2) \\
\geq& max(max(\varphi_1.wpd, \varphi_2.wpd) + u_1, max(\varphi_3.wpd, \varphi_2.wpd) + u_1) \\
=& max(max(\varphi_1.wpd, \varphi_2.wpd), max(\varphi_3.wpd, \varphi_2.wpd)) + u_1 \\
=& max(max(\varphi_1.wpd, \varphi_3.wpd), \varphi_2.wpd) + u_1 \\
=& max(max(\varphi_1.wpd, \varphi_3.wpd), \varphi_2.wpd) + u_3 \\
=& \psi_2.wpd
\end{aligned}$$

Therefore we have $\psi_1 \equiv \psi_2$ by Theorem 1 so that $mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2) \leq mem_{AST}(\varphi) - mem_{AST}(\psi_1)$ by Lemma 2.

Next we show that

$$\begin{aligned}
\psi_1.bpd =& min(min(\varphi_1.bpd, \varphi_2.bpd) + l, min(\varphi_3.bpd, \varphi_2.bpd) + l) \\
=& min(min(\varphi_1.bpd, \varphi_2.bpd), min(\varphi_3.bpd, \varphi_2.bpd)) + l \\
=& min(min(\varphi_1.bpd, \varphi_3.bpd), \varphi_2.bpd) + l \\
=& \psi_2.bpd
\end{aligned}$$

Using Lemma 1 we see that $mem_{node}(\wedge_{\psi_1}) = mem_{node}(\mathcal{U}_{[l,u_3]})$ since $\psi_1.bpd \leq \psi_2.bpd$. Then

$$\begin{aligned}
mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) =& (mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2)) + mem_{AST}(\psi_2) \\
=& (mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2)) + \\
& mem_{node}(\mathcal{U}_{[l,u_3]}) + mem_{node}(\wedge_{\psi_2}) + \\
& mem_{AST}(\varphi_1) + mem_{AST}(\varphi_2) + mem_{AST}(\varphi_3) \\
\leq& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + \\
& mem_{node}(\mathcal{U}_{[l,u_3]}) + mem_{node}(\wedge_{\psi_2}) + \\
& mem_{AST}(\varphi_1) + mem_{AST}(\varphi_2) + mem_{AST}(\varphi_3) \\
=& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + \\
& mem_{node}(\wedge_{\psi_1}) + mem_{node}(\wedge_{\psi_2}) + \\
& mem_{AST}(\varphi_1) + mem_{AST}(\varphi_2) + mem_{AST}(\varphi_3) \\
\leq& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + mem_{node}(\wedge_{\psi_1}) + \\
& mem_{node}(\mathcal{U}_{[l,u_1]}) + mem_{node}(\mathcal{U}_{[l,u_2]}) \\
& mem_{AST}(\varphi_1) + mem_{AST}(\varphi_2) + mem_{AST}(\varphi_3) \\
=& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + mem_{AST}(\psi_1) \\
=& mem_{AST}(\varphi)
\end{aligned}$$

The proof follows for $(\varphi_1 \mathcal{U}_{[l,u_1]} \varphi_2) \vee (\varphi_1 \mathcal{U}_{[l,u_2]} \varphi_3) \rightarrow \varphi_1 \mathcal{U}_{[l,u_1]} (\varphi_2 \vee \varphi_3)$.

(R8): Let $\psi_1 = \varphi \mathcal{U}_{[l,u_1]} \Box_{[0,u_2]} \varphi$, $\psi_2 = \Box_{[l,l+u_2]} \varphi$ where $l \leq u_1$. We show that

$$\begin{aligned}\psi_1.wpd =& max(\varphi.wpd, \varphi.wpd + u_1) + u_2 \\ =& \varphi.wpd + u_1 + u_2 \\ \geq& \varphi.wpd + l + u_2 \\ =& \psi_2.wpd\end{aligned}$$

Therefore we have $\psi_1 \equiv \psi_2$ by Theorem 1 so that

$$mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2) \leq mem_{AST}(\varphi) - mem_{AST}(\psi_1)$$

by Lemma 2. Next we show that

$$\begin{aligned}\psi_1.bpd =& min(\varphi.bpd, \varphi.bpd + 0) + l \\ =& \varphi.bpd + l \\ =& \psi_2.bpd\end{aligned}$$

Using Lemma 1 we see that $mem_{node}(\mathcal{U}_{[l,u_1]}) = mem_{node}(\Box_{[l,l+u_2]})$ since $\psi_1.bpd \leq \psi_2.bpd$. Then

$$\begin{aligned}mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) =& (mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2)) + mem_{AST}(\psi_2) \\ =& (mem_{AST}(\varphi(\psi_1 \mapsto \psi_2)) - mem_{AST}(\psi_2)) + \\ & \quad mem_{node}(\Box_{[l,l+u_2]}) + mem_{node}(\varphi) + \\ \leq& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + \\ & \quad mem_{node}(\Box_{[l,l+u_2]}) + mem_{node}(\varphi) + \\ =& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + \\ & \quad mem_{node}(\mathcal{U}_{[l,u_1]}) + mem_{node}(\varphi) + \\ \leq& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + mem_{node}(\mathcal{U}_{[l,u_1]}) + \\ & \quad mem_{node}(\Box_{[0,u_2]}) + 2 \cdot mem_{node}(\varphi) \\ =& (mem_{AST}(\varphi) - mem_{AST}(\psi_1)) + mem_{AST}(\psi_1) \\ =& mem_{AST}(\varphi)\end{aligned}$$

The proof follows for $\varphi \mathcal{U}_{[l,u_1]} \Diamond_{[0,u_2]} \varphi \to \Diamond_{[l,l+u_2]} \varphi$.

$\Box$