F1Tenth Ego-Vehicle Model Predictive Control

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1 Model Predictive Control

Model Predictive Control (MPC) predicts future states of a system using a given model and a given prediction horizon and minimizes a optimization function as to decrease the error in the future. MPC is superior to other control methods (e.g., PID and LQR) in that it can handle multiple-input multiple-output systems while also considering the system model and a set of given constraints. There are many different implementations of MPC; see [1, 2] for a list of different MPC approaches. We will implement a linear MPC solved through quadratic programming.

1.1 F1Tenth Vehicle Model

A kinematic model is a simple model, while a dynamic model is a more complex model that considers how forces affect motion (e.g., forces on the tires). While a kinematic model is only accurate at small velocities (e.g., less than $\frac{1}{2}\frac{m}{s}$), a kinematic model allows for higher computational throughput compared to a dynamic model, but when experiencing higher velocities the kinematic model introduces high degrees of uncertainty. Although the kinematic model suffers from inaccuracies at large velocities (e.g., greater than $\frac{1}{2}\frac{m}{s}$), a kinematic model is sufficient for our MPC problem since MPC is recalculated at each timestamp.

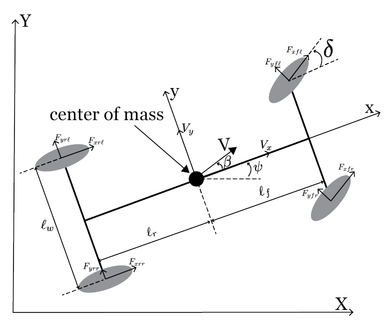

Fig. 1. F1Tenth Model.

Kinematic Non-linear Model. The F1Tenth autonomous racing vehicle is modeled with a kinematic bicycle model for lateral motion modified from [3] as follows:

$$\dot{x} = V cos(\psi + \beta)$$

$$\dot{y} = V sin(\psi + \beta)$$

$$\dot{\psi} = \frac{V cos(\beta)}{\ell_f + \ell_r} tan(\delta)$$

$$\beta = tan^{-1}(\frac{\ell_r tan(\delta)}{\ell_f + \ell_r})$$

where x and y are the x- and y-coordinates of the center of gravity of the vehicle in the global frame, ψ is the angle of the vehicle relative to the x-axis, V is the velocity of the vehicle, β is the slip angle, ℓ_f and ℓ_r are the distances from the center of mass to the front and rear wheels, and δ is the steering angle (Figure 1).

Discrete Model. We discretized the nonlinear system as follows using Euler's forward method:

$$\begin{aligned} x_{k+1} &= x_k + \dot{x} * dt = x_k + V cos(\psi + \beta) * dt \\ y_{k+1} &= y_k + \dot{y} * dt = y_k + V sin(\psi + \beta) * dt \\ \psi_{k+1} &= \psi_k + \dot{\psi} * dt = \psi_k + \frac{V cos(\beta)}{\ell_f + \ell_r} tan(\delta) * dt \end{aligned}$$

Linear Model. A linearized discrete-state model is given in the following form:

$$\mathbf{X}_{k+1} = \mathbf{A}\mathbf{X}_k + \mathbf{B}\mathbf{U}_k + \mathbf{C}$$

where

$$\begin{aligned} \mathbf{X}_k^T &= [x_k, y_k, \psi_k]^T \\ \mathbf{U}_k^T &= [V_k, \delta_k]^T \end{aligned}$$

Using the discretized model, we can now linearize the nonlinear model by performing Taylors' Series expansion around a reference point. We define the reference and equilibrium points as follows:

$$\begin{aligned} \mathbf{X}_{ref} = \mathbf{X}_e &= [x_{ref}, y_{ref}, \psi_{ref}]^T \\ \mathbf{U}_{ref} &= [V_{ref}, \delta_{ref}]^T \\ \mathbf{U}_e &= [0, \delta_{ref}]^T \end{aligned}$$

The Taylors' Series expansion is evaluated as follows:

$$\begin{aligned} x_{k+1} = x_k + &\left[\frac{\partial}{\partial \psi}\bigg|_{\substack{\mathbf{X}=\mathbf{X}_{ref} \\ \mathbf{U}=\mathbf{U}_{ref}}} V cos(\psi + \beta) * dt \right] (\psi_k - \psi_{ref}) + \\ &\left[\frac{\partial}{\partial V}\bigg|_{\substack{\mathbf{X}=\mathbf{X}_{ref} \\ \mathbf{U}=\mathbf{U}_{ref}}} V cos(\psi + \beta) * dt \right] (V_k - 0) \end{aligned}$$

$$\begin{aligned} y_{k+1} = y_k + &\left[\frac{\partial}{\partial \psi}\bigg|_{\substack{\mathbf{X}=\mathbf{X}_{ref} \\ \mathbf{U}=\mathbf{U}_{ref}}} V sin(\psi + \beta) * dt \right] (\psi_k - \psi_{ref}) + \\ &\left[\frac{\partial}{\partial V}\bigg|_{\substack{\mathbf{X}=\mathbf{X}_{ref} \\ \mathbf{U}=\mathbf{U}_{ref}}} V sin(\psi + \beta) * dt \right] (V_k - 0) \end{aligned}$$

$$\psi_{k+1} = \psi_k + \left[\frac{\partial}{\partial V}\bigg|_{\substack{\mathbf{X}=\mathbf{X}_{ref}\\ \mathbf{U}=\mathbf{U}_{ref}}} \frac{V cos(\beta)}{\ell_f + \ell_r} tan(\delta) * dt \right] (V_k - 0) + \left[\frac{\partial}{\partial \delta}\bigg|_{\substack{\mathbf{X}=\mathbf{X}_{ref}\\ \mathbf{U}=\mathbf{U}_{ref}}} \frac{V cos(\beta)}{\ell_f + \ell_r} tan(\delta) * dt \right] (\delta_k - \delta_{ref})$$

The final linear model is as follows:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ \psi_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -V_{ref} sin(\psi_{ref} + \beta_{ref}) dt \\ 0 & 1 & V_{ref} cos(\psi_{ref} + \beta_{ref}) dt \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ \psi_k \end{bmatrix} + \begin{bmatrix} cos(\psi_{ref} + \beta_{ref}) dt & 0 \\ sin(\psi_{ref} + \beta_{ref}) dt & 0 \\ \frac{cos(\beta_{ref})}{\ell_f + \ell_r} tan(\delta_{ref}) dt & \frac{V_{ref} cos(\beta_{ref})}{(\ell_f + \ell_r) cos^2(\delta_{ref})} dt \end{bmatrix} \begin{bmatrix} V_k \\ \delta_k \end{bmatrix} + \begin{bmatrix} V_{ref} \psi_{ref} sin(\psi_{ref} + \beta_{ref}) dt \\ -V_{ref} \psi_{ref} cos(\psi_{ref} + \beta_{ref}) dt \\ \frac{-V_{ref} \delta_{ref} cos(\beta_{ref})}{(\ell_f + \ell_r) cos^2(\delta_{ref})} dt \end{bmatrix}$$

1.2 Optimization Problem.

Our goal with MPC is to minimize the deviation from the reference trajectory (i.e., the track centerline) while satisfying all constraints. Therefore, our objective cost function is given as follows:

$$\min_{\mathbf{X},\mathbf{U}} \sum_{k=0}^{N-1} \{ (\mathbf{X}_k - \mathbf{X}_{ref,k})^T \mathbf{Q} (\mathbf{X}_k - \mathbf{X}_{ref,k}) + \mathbf{U}_k^T \mathbf{R} \mathbf{U}_k \} + (\mathbf{X}_N - \mathbf{X}_{ref,N})^T \mathbf{Q} (\mathbf{X}_N - \mathbf{X}_{ref,N})$$

$$\begin{aligned} \text{such that: } & \mathbf{X}_0 = \text{current state} & \\ & \mathbf{X}_{k+1} = \mathbf{A}_k \mathbf{X}_k + \mathbf{B}_k \mathbf{U}_k + \mathbf{C} & \forall k \in 0, 1, 2, \ldots, N \\ & 0.5 \frac{m}{s} \leq V_k \leq 5.0 \frac{m}{s} & \forall k \in 0, 1, 2, \ldots, N \\ & -25^\circ \leq \delta_k \leq 25^\circ & \forall k \in 0, 1, 2, \ldots, N \end{aligned}$$

where N is the prediction horizon, $\mathbf{X}_k^T = [x_k, y_k, \psi_k]^T$, $\mathbf{U}_k^T = [V_k, \delta_k]^T$, $\mathbf{X}_{ref,k}$ is the reference trajectory, x and y are the x- and y-coordinates of the center of gravity of the vehicle in the global frame, ψ is the angle of the vehicle relative to the x-axis, V is the velocity of the vehicle, δ is the steering angle, $\mathbf{Q}$ is a positive semi-definite weight matrices of size 3x3, and $\mathbf{R}$ is a positive definite weight matrix of size 2x2, and $\mathbf{A}_k$, $\mathbf{B}_k$, and $\mathbf{C}$ are the model parameter matrices defined above. Additionally, we define the reference point as the current point, which is sufficient for our implementation. Our $\mathbf{Q}$ and $\mathbf{R}$ weight matrixes are given as follows:

$$\mathbf{Q} = \begin{bmatrix} 5.0 & 0.0 & 0.0 \\ 0.0 & 5.0 & 0.0 \\ 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 10 & 0.0 \\ 0.0 & 0.1 \end{bmatrix}$$

1.3 Quadratic Programming.

We will use Operator Splitting Quadratic Program (OSQP) to solve for our optimization function [4]. The OSQP solver solves a quadratic programming (QP) in the following form:

$$\min \frac{1}{2} z^T H z + g^T z$$
$$\text{such that: } lb \leq A_c z \leq ub$$

where z is a $nx1$ matrix, H is a nxn matrix, g is a $nx1$ matrix, and A_c is a mxn matrix.

We can rewrite our cost function in Section 1.2 to fit within the OSQP form as follows:

$$z = [\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_N, \mathbf{U}_0, \mathbf{U}_1, \ldots, \mathbf{U}_{N-1}]$$
$$H = diag(\mathbf{Q}, \mathbf{Q}, \mathbf{Q}, \ldots, \mathbf{Q}, \mathbf{R}, \mathbf{R}, \ldots, \mathbf{R})$$

where $\mathbf{Q}$ is repeated N times and and $\mathbf{R}$ is repeated $N-1$ times.

$$g = [-\mathbf{QX}_{ref,0}, -\mathbf{QX}_{ref,1}, \ldots, -\mathbf{QX}_{ref,N-1}, -\mathbf{QX}_{ref,N}, 0, 0, \ldots, 0]^T$$

where 0 is repeated $N-1$ times.

$$lb = [-x, -\mathbf{C}, \ldots, -\mathbf{C}, \mathbf{U}_{min}, \ldots, \mathbf{U}_{min}]^T$$

where $\mathbf{C}$ and $\mathbf{U}_{min}$ are repeated $N-1$ times where $\mathbf{C}$ is a model parameter matrix (defined above) and $\mathbf{U}_{min} = [V_{min}, \delta_{min}]^T = [0.5\frac{m}{s}, -25°]^T$.

$$ub = [-x, -\mathbf{C}, \ldots, -\mathbf{C}, \mathbf{U}_{max}, \ldots, \mathbf{U}_{max}]^T$$

where $\mathbf{C}$ and $\mathbf{U}_{min}$ are repeated $N-1$ times where $\mathbf{C}$ is a model parameter matrix (defined above) and $\mathbf{U}_{max} = [V_{max}, \delta_{max}]^T = [5.0\frac{m}{s}, 25°]^T$.

$$A_c = \left[\begin{array}{ccccc|cccc} -I & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ A & -I & 0 & \ldots & 0 & B & 0 & \ldots & 0 \\ 0 & A & -I & \ldots & 0 & 0 & B & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -I & 0 & 0 & \ldots & 0 \\ \hline 0 & 0 & 0 & \ldots & 0 & I & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & I \end{array}\right]$$

where A and B are model parameter matrices (defined above) and I is an identity matrix of size 3x3.

References

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