Mission-time LTL (MLTL) Formula Validation Via Regular Expressions [⋆]

Jenna Elwing[1]*, Laura Gamboa-Guzman[2][0000−0003−3133−5148], Jeremy Sorkin[3]*, Chiara Travesset[4]*, ✉ Zili Wang[2][0000−0003−1730−6180]*, and Kristin Yvonne Rozier[2][0000−0002−6718−2828]

[1] University of Michigan, Ann Arbor, MI 48109, USA
jelwing@umich.edu
[2] Iowa State University, Ames, IA 50011, USA
lpgamboa@iastate.edu,✉ziliw1@iastate.edu,
kyrozier@iastate.edu
[3] University of Amsterdam, 1018 WV Amsterdam, Netherlands
jeremy.sorkin@student.uva.nl
[4] University of Wisconsin, Madison, WI 53715, USA
travesset@wisc.edu

Abstract. Mission-time Linear Temporal Logic (MLTL) represents the most practical fragment of Metric Temporal Logic; MLTL resembles the popular logic Linear Temporal Logic (LTL) with finite closed-interval integer bounds on the temporal operators. Increasingly, many tools reason over MLTL specifications, yet these tools are useful only when system designers can validate the input specifications. We design an automated characterization of the structure of the computations that satisfy a given MLTL formula using regular expressions. We prove soundness and completeness of our structure. We also give an algorithm for automated MLTL formula validation and analyze its complexity both theoretically and experimentally. Additionally, we generate a test suite using control flow diagrams to robustly test our implementation and release an open-source tool with a user-friendly graphical interface. The result of our contributions are improvements to existing algorithms for MLTL analysis, and are applicable to many other tools for automated, efficient MLTL formula validation. Our updated tool may be found at https://temporallogic.org/research/WEST.

Keywords: Mission-time Linear Temporal Logic (MLTL) · MLTL Validation · Temporal Logic Validation.

1 Introduction

System specifications, such as aerospace operational concepts, often utilize timelines to express critical requirements. We can cite examples of this from NASA's Automated Airspace Concept [11], the U.S. Navy's Aircraft Carrier Deck Scheduler [33], the JAXA-NASA Global Precipitation Measurement (GPM) Observatory [10], and many

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[*] These authors contributed equally to this work.

others. Formal methods provide continuously advancing tools and techniques to rigorously analyze timelines expressed in the form of temporal logic requirements, from early design-time model checking and theorem proving to on-board runtime verification. The U. S. Federal Aviation Administration (FAA) even advocates the use of formal methods for flight certification of these critical systems [28,29,27]. Yet, a significant hurdle to the use of formal methods remains: how to convincingly demonstrate to the humans in the loop, from system designers to certifiers, that the analyzed formulas truly represent the system requirements [31]. We creatively address this validation question using regular expressions.

NASA, for example, has developed several tools that operate over temporal logic requirements, such as FRET [12], R2U2 [32], and a PVS library [7] for the logic MLTL (Mission-time Linear Temporal Logic) [30,19]. MLTL was the specification logic for NASA's Robonaut2 verification project [16] and is currently the specification logic for both design-time and runtime verification of the NASA Lunar Gateway Vehicle System Manager [8]. Other recent verification efforts involving MLTL include a JAXA autonomous satellite [24], a UAS Traffic Management (UTM) system involving Collins and Mosaic Aerospace [13], a sounding rocket [14], and multiple small satellites [21,20,2]. However, all of these successful verification efforts were carried out by groups specializing in formal methods research. To enable broader application of formal verification, and adoption across larger projects, we critically need better validation, e.g., so that analysis over MLTL-specified requirements can transparently contribute to flight certification.

Many specifications from case studies, in logics such as Metric Temporal Logic (MTL) [1] and Signal Temporal Logic (STL) [22], fall within the MLTL fragments of these logics. Variations on MTL such as MLTL have grown increasingly popular, in part due to their comparatively tractable complexity-to-expressibility trade-offs [25]. The model checker nuXmv encodes a popular subset of MLTL for use in symbolic model checking [17].

There exists a SAT solver for MLTL, MLTLSAT [19], but there are currently no tools for MLTL formula validation. This paper introduces the WEST tool [GitHub][5] repository, which produces a description of the set of all finite timelines (of a fixed length) that satisfy a given MLTL formula, similar to a truth table for propositional formulas. MLTL validation can be done by verifying that the output of the WEST program indeed matches the behaviour of the specification in question.

We show that our contributions not only fill a critical gap in temporal logic validation, but also directly connect to parallel developments to enable better temporal logic formula analysis, benchmark generation, proof generation (e.g., in ACL2), and synthesis of verified C++ code from temporal logic behavior descriptions.

We structure the paper as follows. Section 2 builds on the semantics of MLTL to define a computation and its bit string representation. Section 3 recursively defines regular expressions encapsulating the satisfying computations of MLTL formulas. We provide a calculation for the minimum computation length required to describe all the satisfying computations of an MLTL formula that slightly improves upon existing calculations in the literature. Finally, we show an application of the regular expressions by using

[5] https://github.com/zwang271/WEST

them to prove an MLTL rewriting theorem. We introduce the WEST tool that implements automated validation in Section 4 and calculate its space and time complexity, both theoretically and experimentally. Section 5 proves the correctness of WEST and provides a test suite to show correctness of implementation with high confidence. Intelligent fuzzing techniques contribute to test suite construction from a state diagram representing the control flow of WEST. We also verify the correctness of outputs of the WEST program against a naïve brute force implementation. Section 6 provides a combinatorial theorem for simplifying certain outputs of the WEST program to the trivial computation. Section 7 demonstrates a specific use case of the WEST tool and explores the currently supported features. Section 8 discusses impacts and future work.

2 Preliminaries: Mission-time LTL and Bit String Computations

Mission-time Linear Temporal Logic (MLTL) [19] is a finite variation of LTL over bounded, closed, discrete intervals of the form $[a, b]$ where $a, b \in \mathbb{N}$ and $0 \leqslant a \leqslant b$. The syntax of MLTL formulas, φ and ψ over a (finite) set of atomic propositions $\mathcal{AP}$, where $p \in \mathcal{AP}$ is a propositional variable, is given by the following BNF grammar:

$$\varphi, \psi := \top \mid \bot \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \mathcal{F}_{[a,b]}\varphi \mid \mathcal{G}_{[a,b]}\varphi \mid \varphi\mathcal{U}_{[a,b]}\psi \mid \varphi\mathcal{R}_{[a,b]}\psi.^{6}$$

The symbols $\mathcal{F}, \mathcal{G}, \mathcal{U}, \mathcal{R}$ respectively denote the temporal operators Finally, Globally, Until, and Release. MLTL formulas can be interpreted using both finite and infinite "timelines" that are called *computations*, which represent a discrete sequence of time instances and the truth values for the propositional variables on each one of these. For the purpose of this paper, we are only going to deal with *finite* computations that represent only finitely many time steps.

Definition 1 (***Finite computations***)**.** *A computation π of length m is a sequence $\{\pi[i]\}_{i=0}^{m-1}$ of sets of propositional variables, $\pi[i] \subseteq \mathcal{AP}$, where the i^{th} set contains the propositional variables that are true at the i^{th} time step. That is, a propositional variable p is true at time step i if and only if $p \in \pi[i]$. We denote the suffix of π starting at i (including i) by π_i. Note that $\pi_0 = \pi$.*

We provide the formal semantics for MLTL below. A computation π satisfies a given MLTL formula α, written $\pi \models \alpha$, in the following cases[7]:

$\pi \models p$ iff $p \in \pi[0]$ $\qquad$ $\pi \models \neg\alpha$ iff $\pi \not\models \alpha$

$\pi \models \alpha \wedge \beta$ iff $\pi \models \alpha$ and $\pi \models \beta$ $\qquad$ $\pi \models \alpha \vee \beta$ iff $\pi \models \alpha$ or $\pi \models \beta$

$\pi \models \mathcal{F}_{[a,b]}\alpha$ iff $|\pi| > a$ and $\exists i \in [a, b]$ such that $\pi_i \models \alpha$

$\pi \models \mathcal{G}_{[a,b]}\alpha$ iff $|\pi| \leqslant a$ or $\forall i \in [a, b]$ $\pi_i \models \alpha$

[6] For simplicity, we do not include parentheses in the grammar, but the WEST program requires parentheses (see Section 4). We encode Release directly rather than as the dual of Until. Refer to the WEST Appendix[9] for a straightforward proof of equivalence using the semantics.

[7] We do not include the Next operator, which is often denoted $\mathcal{X}$, since it is equivalent to both $\mathcal{G}_{[1,1]}$ and $\mathcal{F}_{[1,1]}$.

$\pi \models \alpha\, \mathcal{U}_{[a,b]}\beta$ iff $|\pi| > a$ and $\exists i \in [a,b]$ such that $\pi_i \models \beta$ and $\forall j \in [a, i-1]$ $\pi_j \models \alpha$

$\pi \models \alpha\, \mathcal{R}_{[a,b]}\beta$ iff $|\pi| \leqslant a$ or $\forall i \in [a,b]$ $\pi_i \models \beta$ or $\exists j \in [a, b-1]$ such that $\pi_j \models \alpha$ and $\forall a \leqslant k \leqslant j$ $\pi_k \models \beta$

Definition 2 (***Bit String Representation of a Computation***)**.** *Let $p_0, p_1, ..., p_{n-1}$ be propositional variables for a fixed $n \in \mathbb{N}$. We represent a (finite) computation π of length $m \in \mathbb{N}$ using a bit string representation as follows:*

- *Each time step $j \in [0, 1, \ldots, m-1]$ corresponds to a bit string of length n, where the k^{th} bit represents the truth assignment of the proposition p_{k-1}.*
- *Each time step is separated by a comma and orders the time steps chronologically.*

Example 1. Suppose $n = 2$. The bit string $\pi = 10, 01$ represents a timeline on which p_0 is true and p_1 is false in the zeroth time step, whereas p_0 is false and p_1 true in the first time step.

3 MLTL Regular Expressions

We modify the standard definition of *Regular Expressions* (regex) to introduce notation that describes the satisfying computations of an MLTL formula. We begin by quoting the parts of the standard definition of a regex from [34] that we use to describe our computations:

Definition 3 (***Regular Expression***)**.** *Let Σ denote an alphabet. We say that R is a regular expression if one of the following holds:*

- *$R = a$, for some $a \in \Sigma$.*
- *$R = \epsilon$, where ϵ is the language containing the empty string.*
- *$R = \emptyset$, the empty set.*
- *$R = (R_1 \vee R_2)$, where R_1, R_2 are regexes and $\vee$ denotes alternation; the set union of all the strings described by R_1 and R_2.*
- *$R = R_1R_2$, which denotes concatenation, i.e., the set of strings obtained by concatenating any string generated by R_1 with any string generated by R_2, in that order.*

Now we introduce the additions to the definition of a regex that we utilize to describe the computations. These additions allow us to write regexes of a known, finite, fixed length, which is required when describing the computations in MLTL.

Definition 4 (***Temporal Regular Expression***)**.** *Let R and T denote regular expressions, and let S be an abbreviation for $(0 \vee 1)$. Let fixed $n \in \mathbb{N}$ denote the number of propositional variables in an MLTL formula. We use the following operations to describe the form of satisfying computations of the formula in the bit string representation:*

- *Pad(R, T) determines which regular expression is longer and concatenates $(, S^n)$ repeatedly to the end of the shorter regular expression until the two regular expressions are the same length. Note that in the bit string representation, $(, S^n)$ denotes a time step in which the truth values of all n propositional variables do not matter.*

- *$R \wedge T$ is the intersection of the sets of strings described by R and T. To perform this operation, we first use Pad(R, T), and then take the set intersection of the sets of strings described by the two regular expressions.*
- *R^i denotes regular expression consisting of R repeated i times for $i \geqslant 0$. $R^0 = \epsilon$.*

Note that our regular expressions do not use the Kleene star. This is because our computations are of a fixed, finite length.

Example 2. Let $n = 2$, and let $R = S1$ and $T = (1S, 1S) \vee (S1, S1)$. To compute $R \wedge T$ and $R \vee T$, we perform Pad(R,T). Since T is the longer regex by one time step, we extend R by one time step. Thus $T = (1S, 1S) \vee (S1, S1)$ and $R = S1, SS$. Now we can perform set intersection and alternation on the two regular expressions of equal length:
- $R \wedge T = (11, 1S) \vee (S1, S1)$
- $R \vee T = (S1, SS) \vee (1S, 1S) \vee (S1, S1) = (S1, SS) \vee (1S, 1S)$

Definition 5 **(*MLTL Regular Expressions*).** *Let $\Sigma = \{$"0" , "1", ","$\}$ be the alphabet and define S as an abbreviation for $(0 \vee 1)$. Let φ and ψ be well-formed MLTL formulas in negation normal form (NNF[8]) containing the n propositional variables $p_0, p_1, ..., p_{n-1}$. We recursively define the regular expression of all satisfying computations for an MLTL formula as follows:*

$$\mathit{reg}(\top) = S^n \qquad \mathit{reg}(\bot) = \varnothing$$

$$\mathit{reg}(p_k) = S^k 1 S^{n-k-1} \qquad \mathit{reg}(\neg p_k) = S^k 0 S^{n-k-1}$$

$$\mathit{reg}(\varphi \vee \psi) = \mathit{reg}(\varphi) \vee \mathit{reg}(\psi) \qquad \mathit{reg}(\varphi \wedge \psi) = \mathit{reg}(\varphi) \wedge \mathit{reg}(\psi)$$

$$\mathit{reg}(\mathcal{G}_{[a,b]}\varphi) = \bigwedge_{i=a}^{b} (S^n,)^i \mathit{reg}(\varphi) \qquad \mathit{reg}(\mathcal{F}_{[a,b]}\varphi) = \bigvee_{i=a}^{b} (S^n,)^i \mathit{reg}(\varphi)$$

$$\mathit{reg}(\varphi\, \mathcal{U}_{[a,b]}\psi) = \bigvee_{i=a}^{b} \mathit{reg}\left(\mathcal{G}_{[a,i-1]}\varphi \wedge \mathcal{G}_{[i,i]}\psi\right)$$

$$\mathit{reg}(\varphi \mathcal{R}_{[a,b]}\psi) = \mathit{reg}\left(\mathcal{G}_{[a,b]}\psi\right) \vee \bigvee_{i=a}^{b-1} \mathit{reg}\left(\mathcal{G}_{[a,i]}\psi \wedge \mathcal{G}_{[i,i]}\varphi\right)$$

Definition 6 **(*Computation Length*).** *We recursively define the computation length $cplen(\varphi)$ of an MLTL formula φ:*

$$\text{cplen}(p_k) = \text{cplen}(\neg p_k) = 1,$$
$$\text{cplen}(\varphi \wedge \psi) = \text{cplen}(\varphi \vee \psi) = \max(\text{cplen}(\varphi), \text{cplen}(\psi)),$$
$$\text{cplen}(\mathcal{G}_{[a,b]}\varphi) = \text{cplen}(\mathcal{F}_{[a,b]}\varphi) = b + \text{cplen}(\varphi),$$
$$\text{cplen}(\varphi\mathcal{U}_{[a,b]}\psi) = \text{cplen}(\varphi\mathcal{R}_{[a,b]}\psi) = b + \max(\text{cplen}(\varphi) - 1, \text{cplen}(\psi)).$$

Here, cplen(φ) is the minimum computation length required to ensure that none of the intervals in φ are out of bounds. A computation that is of length cplen(φ) or greater

[8] Note that any MLTL formula can easily be converted into NNF.

will reach the end of every interval in φ. Our minimum computation length for Until and Release are slight optimizations of what was previously considered the minimum computation length in the literature. The previous bound in [16] was

$$\text{cplen}(\varphi\mathcal{U}_{[a,b]}\psi) = \text{cplen}(\varphi\mathcal{R}_{[a,b]}\psi) = b + \max(\text{cplen}(\varphi), \text{cplen}(\psi))$$

whereas Theorem 1 proves our minimum computation length for Until and Release is

$$\text{cplen}(\varphi\mathcal{U}_{[a,b]}\psi) = \text{cplen}(\varphi\mathcal{R}_{[a,b]}\psi) = b + \max(\text{cplen}(\varphi) - \mathbf{1}, \text{cplen}(\psi)).$$

Theorem 1 (***Minimum Computation Length of Until and Release***)**.** *Let $0 \leqslant a \leqslant b \in \mathbb{N}$ and let φ, ψ be well-formed MLTL formulas in NNF. The minimum computation length of $\mathcal{R}$ and $\mathcal{U}$ is given by $cplen(\varphi\mathcal{U}_{[a,b]}\psi) = cplen(\varphi\mathcal{R}_{[a,b]}\psi) = b + max(cplen(\varphi) - 1, cplen(\psi))$.*

The formulas for minimum computation length follow directly from the regular expressions for the satisfying computations for Until and Release and the minimum computation lengths for Finally, Globally, AND, and OR. See the WEST Appendix[9] for details of the proof.

We can reduce the minimum computation length of Until for the formula $\varphi\mathcal{U}_{[a,b]}\psi$ for the following reason: ψ must be assigned true at time step b if it has not been true at a prior time step by the semantics of $\mathcal{U}$, and thus the truth value of φ at time step b does not matter. Likewise for the formula $\varphi\mathcal{R}_{[a,b]}\psi$, if ψ is true from time step a to time step b, the computation satisfies the formula regardless of the value of φ at time step b.

Let $\mathscr{L}(\text{reg}(\varphi))$ denote the language of reg(φ), i.e., the set of computations represented by the regular expression reg(φ).

Theorem 2 (***Soundness and Completeness***)**.** *For any well-formed MLTL formula φ in negation normal form, a computation π with $|\pi| = cplen(\varphi)$ satisfies φ if and only if $\pi \in \mathscr{L}(\text{reg}(\varphi))$.*

We omit the proof for this theorem since it follows straightforward by induction on the length of a formula. See the WEST Appendix [9] for details of the proof. As an application of the regular expressions in the above theorem, we prove a previously known MLTL rewriting theorem. This demonstrates the utility of our regular expressions for theoretical analysis.

Theorem 3 (***Nested Until and Release Rewriting Theorem***)**.** *Any MLTL formula using the Until or Release operator can be rewritten with right-nested subformulas. Let $a, b, c \in \mathbb{Z}_{\geqslant 0}$, $a \leqslant b$, and φ, ψ be well-formed MLTL formulas in NNF. Then $\varphi\,\mathcal{U}_{[a,b+c]}\psi \equiv \varphi\,\mathcal{U}_{[a,b]}(\varphi\,\mathcal{U}_{[0,c]}\psi)$ and $\varphi\,\mathcal{R}_{[a,b+c]}\psi \equiv \varphi\,\mathcal{R}_{[a,b]}(\varphi\,\mathcal{R}_{[0,c]}\psi)$.*

The proof is omitted here as it follows from the definition of regular expressions for MLTL. See the WEST Appendix[9] for details of the proof.

4 WEST Algorithm and Analysis

Algorithm 1 (Fig. 1) recursively computes all satisfying temporal regular expressions to an input formula φ. We use sets to represent alternation of regular expressions; for n regular expressions $t_0, ..., t_{n-1}$, we write $\{t_0, ..., t_{n-1}\} = \bigcup_{i=0}^{n-1} t_i$. Additionally, we

[9] The Appendix for this paper can be found at https://temporallogic.org/research/WEST .

provide details for performing set intersection of temporal regular expressions, and the algorithms for the temporal operators $\mathcal{G}$ and $\mathcal{U}$. Note how `reg_U` parallels the regular expression defined for $\mathcal{U}$, and the algorithms (see WEST Appendix[9]) for the other three temporal operators follow an identical structure.

For regular expressions w_0 and w_1, a useful reduction is that $\{w_0 1 w_1, w_0 0 w_1\} = \{w_0 s w_1\}$. Each time we perform set intersection, we greedily apply this reduction to all appropriate pairs of regular expressions in the set. This prevents repeated set intersection operations from blowing up exponentially most of the time, and drastically improves running time. We call this simple algorithm `simplify` and use it extensively throughout the WEST code.

4.1 Proof of Correctness of WEST

Theorem 4 (***Theoretical Correctness of WEST***)**.** *Given a well-formed MLTL formula, the WEST Algorithm outputs the regular expressions of the satisfying computations as described in Section 3.*

Proof. Correctness of the WEST algorithm is dependent on the correctness of subroutines `reg_prop_cons`, `reg_prop_var`, `join`, `set_intersect`, `reg_F`, `reg_G`, `reg_U`, and `reg_R`. The routines `reg_prop_cons` and `reg_prop_var` take as input an MLTL formula of the appropriate form and return the regular expression defined in Section 3.

The function `join` concatenates two sets of regular expressions R and T, which is equivalent to $\mathscr{L}(R) \cup \mathscr{L}(T)$. `set_intersect` takes as input two sets of regular expressions $R = \{r_0, ..., r_{a-1}\}$ and $T = \{t_0, ..., t_{b-1}\}$, such that each r_i and t_j are regular expressions over $\Sigma = \{$“0” , “1”, “S”, “,”$\}$. Without lost of generality, assume that all strings of regular expressions are right-padded to equal length. We show that $\mathscr{L}(\texttt{set\_intersect}(R,T)) = \mathscr{L}(R) \cap \mathscr{L}(T)$:

$$\mathscr{L}(R) \cap \mathscr{L}(T) = \left(\bigcup_{i=0}^{a-1} \mathscr{L}(r_i)\right) \cap \left(\bigcup_{j=0}^{b-1} \mathscr{L}(t_j)\right) = \bigcup_{i=0}^{a-1} \bigcup_{j=0}^{b-1} \left(\mathscr{L}(r_i) \cap \mathscr{L}(t_j)\right).$$

The loop in `set_intersect` computes the union of $\texttt{bit\_wise\_and}(r_i, t_j)$ over all such pairs, and so it suffices to show $\mathscr{L}(\texttt{bit\_wise\_and}(r_i, t_j)) = \mathscr{L}(r_i) \cap \mathscr{L}(t_j)$. Given a computation π, $\pi \in \mathscr{L}(r_i) \cap \mathscr{L}(t_j)$ if and only if π matches every character of both r_i and t_j. $\texttt{Bit\_wise\_and}(r_i, t_j)$ compares r_i and t_j character by character and computes their intersection, which is defined naturally: $0 \cap 1 = \varnothing$, $0 \cap S = 0$, $1 \cap S = 1$, $0 \cap 0 = 0$, $1 \cap 1 = 1$, and $S \cap S = S$. Note that this operation is commutative. This exhaustively captures all the cases for which π must match corresponding characters from r_i and t_j. Thus $\mathscr{L}(\texttt{bit\_wise\_and}(r_i, t_j)) = \mathscr{L}(r_i) \cap \mathscr{L}(t_j)$ and the claim holds.

The correctness for `reg_F`, `reg_G`, `reg_U`, `reg_R`, and `reg` is proven by induction on depth of recursion to `reg`. The depth of recursion is exactly the depth of the parse tree of the input formula. For the base case (depth 0), `reg_prop_var` and `reg_prop_cons` are called to handle input formulas that consist of a propositional variable, the negation of a propositional variable, or a propositional constant. Then assume `reg` is correct on all formulas of depth at most d, for some integer $d \geqslant 0$. Let γ be an MLTL formula in negation normal form of depth $d + 1$. Then γ must be of

Algorithm 1 WEST Algorithm

```
Inputs: $\varphi$ - MLTL formula in NNF
$\varphi_1$ and $\varphi_2$ below are subformulas of $\varphi$
$n$ - number of propositional variables
Output: set of REGEX satisfying $\varphi$
 1: procedure REG(string $\varphi$, int $n$)
 2:     if $\varphi$ is $\top$ or $\bot$ then
 3:         return reg_prop_const($\varphi$, n)
 4:     if $\varphi$ is $p_k$ or $\neg p_k$ then
 5:         return reg_prop_var($\varphi$, n)
 6:     if $\varphi = \varphi_1 \wedge \varphi_2$ then
 7:         return set_intersect(reg($\varphi_1$), reg($\varphi_2$), n)
 8:     if $\varphi = \varphi_1 \vee \varphi_2$ then
 9:         return join(reg($\varphi_1$), reg($\varphi_2$), n)
10:     if $\varphi = \mathcal{F}_{[a,b]}\varphi_1$ then
11:         return reg_F(reg($\varphi_1$), a, b, n)
12:     if $\varphi = \mathcal{G}_{[a,b]}\varphi_1$ then
13:         return reg_G(reg($\varphi_1$), a, b, n)
14:     if $\varphi = \varphi_1\mathcal{U}_{[a,b]}\varphi_2$ then
15:         return reg_U(reg($\varphi_1$), reg($\varphi_2$), a, b, n)
16:     if $\varphi = \varphi_1\mathcal{R}_{[a,b]}\varphi_2$ then
17:         return reg_R(reg($\varphi_1$), reg($\varphi_2$), a, b, n)
```

Algorithm 2 set_intersect

```
Inputs: $R, T$ - two sets of REGEX
$n$ - number of propositional variables
Output: set of REGEX equal to $R \wedge T$
 1: procedure SET_INTERSECT($R$, $T$, n)
 2:     Pad($R$, $T$, n), ret $\leftarrow$ {}
 3:     for $(r, t) \in R \times T$ do
 4:         add bit_wise_and($r$, $t$) to ret
 5:     return simplify(ret)
```

Algorithm 3 reg_G

```
Inputs: $r_\varphi$ - set of REGEX for MLTL formula $\varphi$ (after calling reg)
$a, b$ - interval bounds
$n$ - number of propositional variables
Output: set of REGEX for $G_{[a,b]}\varphi$
 1: procedure REG_G(set $r_\varphi$, int $a$, int $b$, int $n$)
 2:     pre $\leftarrow$ ((‘$S$’)$^n$ + ‘,’)$^a$
 3:     comp $\leftarrow r_\varphi$
 4:     if $a > b$ then return $\{S^n\}$
 5:     for $(1 \leqslant i \leqslant b - a)$ do
 6:         temp$_\varphi \leftarrow$ ((‘$S$’)$^n$ + ‘,’)$^i$ + $r_\varphi$
 7:         comp $\leftarrow$ set_intersect(comp, temp$_\varphi$, n)
 8:     return pre + comp
```

Algorithm 4 reg_U

```
Inputs: $r_\varphi$, $r_\psi$ - sets of REGEX for MLTL formulas $\varphi$ and $\psi$ (after calling reg)
$a, b$ - integers representing interval bound
$n$ - number of propositional variables
Output: set of REGEX for $\varphi\mathcal{U}_{[a,b]}\psi$
 1: procedure REG_U($r_\varphi$, $r_\psi$, $a$, $b$, $n$)
 2:     comp $\leftarrow$ ((‘$S$’)$^n$ + ‘,’)$^a$+ $r_\psi$
 3:     if $a > b$ then return {}
 4:     for $(a \leqslant i \leqslant b - 1)$ do
 5:         G1 $\leftarrow$ reg_G($r_\varphi$, $a$, $i$, $n$)
 6:         G2 $\leftarrow$ reg_G($r_\psi$, $i+1$, $i+1$, $n$)
 7:         temp_comp $\leftarrow$ set_intersect(G1, G2, n)
 8:         comp $\leftarrow$ join( comp, temp_comp)
 9:     return comp
```

Fig. 1: Pseudocode for `WEST`, `set_intersect`, `reg_G`, and `reg_U`. The pseudocode for all other algorithms referenced can be found in the WEST Appendix[9].

the form $\varphi \vee \psi$, $\varphi \wedge \psi$, $\mathcal{G}_{[a,b]}\varphi$, $\mathcal{F}_{[a,b]}\varphi$, $\varphi\mathcal{U}_{[a,b]}\psi$, or $\varphi\mathcal{R}_{[a,b]}\psi$, for some formulas φ and ψ of depth at most d, and a pair of non-negative integers, a and b. Correctness of the first two cases have been proven. The proof for the four temporal cases are of similar structure, and it suffices to verify that the algorithms compute appropriate regular expressions correctly using `join` and `set_intersect`.

We give the explicit proof for the case $\gamma = \varphi\mathcal{U}_{[a,b]}\psi$ as an example. `reg_U` takes as input $r_\varphi = \text{reg}(\varphi)$ and $r_\psi = \text{reg}(\psi)$, which by the induction hypothesis are correctly computed. The regular expression for the Until operator may be rewritten as $\text{reg}(\varphi\ \mathcal{U}_{[a,b]}\psi) = \text{reg}\left(\mathcal{G}_{[a,a]}\psi\right) \vee \bigvee_{i=a}^{b-1} \text{reg}\left(\mathcal{G}_{[a,i]}\varphi \wedge \mathcal{G}_{[i+1,i+1]}\psi\right)$. In line 2 of reg_U, the variable comp is initialized to $(\text{“}S\text{”}^n + \text{“},\text{”})^a$ pre-concatenated to r_ψ, and is the regular expression for $\mathcal{G}_{[a,a]}\psi$. Next, the $\vee$ from $i = a$ to $b - 1$ is computed by the for loop in line 4. Lastly, lines 5 through 7 computes reg $\left(\mathcal{G}_{[a,i]}\varphi \wedge \mathcal{G}_{[i+1,i+1]}\psi\right)$. This shows correctness of reg_U; although, in a complete proof, correctness of reg_G needs to be shown first since lines 5 and 6 calls reg_G. Continuing in the same fashion to prove the other three cases, reg is correct on all depth $d + 1$ inputs, and thus reg is correct on all inputs by induction.

4.2 Theoretical Complexity

In order to reason about the complexity of our algorithms, we first introduce several assumptions about the input. Suppose that the lower and upper intervals of temporal operators are bounded by some constant $d \in \mathbb{N}$, and that the difference between any bound is less than some constant $\delta \in \mathbb{N}$. These are reasonable assumptions since MLTL turns into a finite temporal logic when a known mission end is given. We provide a summary of the complexity of each of the operators that contribute to the worst-case behavior of the final output.

For any function $f(\varphi)$ taking a string argument φ, we use $|\varphi|$ to denote the number of characters in φ and $S(f(\varphi))$ to denote the space complexity of f in terms of the number of characters in the output.

If φ is a propositional constant or variable, it is easy to see that $S(\texttt{reg}(\perp)) = 0$ since only the empty set is returned. By definition, $\texttt{reg}(\top) = S^n$, so we have that $S(\texttt{reg}(\top)) = n$. Similarly, $\texttt{reg}(p_k)$ and $\texttt{reg}(\neg p_k)$ both return strings of length n, whence $S(\texttt{reg}(p_k)) = S(\texttt{reg}(\neg p_k)) = S(\texttt{reg}(\top)) = n$.

If φ is “$\varphi_1 \vee \varphi_2$”, we return $\texttt{join}(\texttt{reg}(\varphi_1), \texttt{reg}(\varphi_2))$, which simply computes the union of the two sets. Thus $S(\texttt{reg}(\varphi_1 \vee \varphi_2)) = S(\texttt{reg}(\varphi_1)) + S(\texttt{reg}(\varphi_2))$.

If φ is “$\varphi_1 \wedge \varphi_2$”, $\texttt{set\_intersect}(\texttt{reg}(\varphi_1), \texttt{reg}(\varphi_2), n)$ returns a set of size $S(\texttt{reg}(\varphi_1)) \cdot S(\texttt{reg}(\varphi_2))$ in the worst case when no simplification can be made. Thus, our space complexity is $S(\texttt{reg}(\varphi_1 \wedge \varphi_2)) = S(\texttt{reg}(\varphi_1)) \cdot S(\texttt{reg}(\varphi_2))$.

For the next cases, we use these two bounds and define the constants $\mathcal{C}_G$ and $\mathcal{C}_F$:

$$\textstyle\prod_{i=a}^{b}(n+1)i = (n+1)^{b-a} \cdot \frac{b!}{(a-1)!} \leqslant (n+1)^{\delta} b! \leqslant (n+1)^{\delta} d! = \mathcal{C}_G$$
$$\textstyle\sum_{i=a}^{b}(n+1)i \leqslant (n+1)b\delta \leqslant (n+1)d\delta = \mathcal{C}_F$$

If φ is “$\mathcal{G}_{[a,b]}\varphi_1$", recall that $\texttt{reg}(\mathcal{G}_{[a,b]}\varphi_1) = \bigwedge_{i=a}^{b}(S^n,)^i\texttt{reg}(\varphi_1)$. From the analysis of set_intersect, worst-case space complexity is multiplicative. Thus $S(\texttt{reg}(\varphi)) = \prod_{i=a}^{b}(n+1)i \cdot S(\texttt{reg}(\varphi_1)) < \mathcal{C}_G \cdot S(\texttt{reg}(\varphi_1))^{\delta}$. In this calculation, $(n+1)i$ counts the concatenation of the padded components in the computation.

If φ is “$\mathcal{F}_{[a,b]}\varphi_1$", recall that $\texttt{reg}(\mathcal{F}_{[a,b]}\varphi_1) = \bigvee_{i=a}^{b}(S^n,)^i\texttt{reg}(\varphi_1)$.
From the analysis of join, worst-case space complexity is additive, which implies that $S(\texttt{reg}(\varphi)) = \sum_{i=a}^{b}(n+1)i \cdot S(\texttt{reg}(\varphi_1)) < \mathcal{C}_F \cdot S(\texttt{reg}(\varphi_1))$.

If φ = “$\varphi_1\mathcal{U}_{[a,b]}\varphi_2$”, then $\texttt{reg}(\varphi_1\,\mathcal{U}_{[a,b]}\varphi_2) = \bigvee_{i=a}^{b} \texttt{reg}\left(\mathcal{G}_{[a,i-1]}\varphi_1 \wedge \mathcal{G}_{[i,i]}\varphi_2\right)$. We can bound $S(\texttt{reg}(\mathcal{G}_{[i,i]}\varphi_2))$ by $(n+1) \cdot i \cdot S(\texttt{reg}(\varphi_2))$ because the operation is equivalent to simply prepending $(S^n,)^i$. Thus, using our previous results for the $\mathcal{G}$, $\wedge$, and $\vee$ operators, we have that:

$$\begin{aligned}
S(\texttt{reg}(\varphi)) &\leqslant \sum_{i=a}^{b}[\mathcal{C}_G S(\texttt{reg}(\varphi_1))^{\delta}\cdot(n+1)iS(\texttt{reg}(\varphi_2))] \\
&\leqslant \delta[\mathcal{C}_G\cdot\delta(n+1)d\cdot S(\texttt{reg}(\varphi_1))^{\delta}S(\texttt{reg}(\varphi_2))] \\
&= \mathcal{C}_U\cdot S(\texttt{reg}(\varphi_1))^{\delta}S(\texttt{reg}(\varphi_2))
\end{aligned}$$

where $\mathcal{C}_U = \mathcal{C}_G\delta(n+1)d$.

If φ = "$\varphi_1\mathcal{R}_{[a,b]}\varphi_2$", recall that
$\texttt{reg}(\varphi_1\mathcal{R}_{[a,b]}\varphi_2) = \texttt{reg}\left(\mathcal{G}_{[a,b]}\varphi_2\right)\vee\bigvee_{i=a}^{b-1}\texttt{reg}\left(\mathcal{G}_{[a,i]}\varphi_2\wedge\mathcal{G}_{[i,i]}\varphi_1\right)$.
A similar argument to the $\mathcal{U}$ case shows $S(\texttt{reg}(\varphi)) < \mathcal{C}_R\cdot S(\texttt{reg}(\varphi_1))\cdot S(\texttt{reg}(\varphi_2))^{\delta}$, where $\mathcal{C}_R = \mathcal{C}_G\cdot(1+\delta(n+1)d)$.

Theorem 5 (***Space Complexity***)**.** *Given a well-formed MLTL formula φ, $\texttt{reg}(\varphi)$ has worst-case space complexity that is $O(\mathcal{C}_R^{\delta^{\ell}}\cdot(\ell+1)^{\delta^{\ell+1}})$, where ℓ is the number of logical connectives in $\{\wedge,\vee,\mathcal{F},\mathcal{G},\mathcal{R},\mathcal{U}\}$ that occurr in φ.*

Proof. To analyze worst-case complexity, it is clear from the analysis above that $\mathcal{U}$ and $\mathcal{R}$ give the worst complexity. In the previous analysis, we defined $\mathcal{C}_U = \mathcal{C}_G\delta(n+1)d$ and $\mathcal{C}_R = \mathcal{C}_G\cdot(1+\delta(n+1)d)$. Observe that $\mathcal{C}_R > \mathcal{C}_U$, and so we analyze only repeated nesting of the operator $\mathcal{R}$.

However, notice that the structure of the parse tree is important. Formulas similar to $(p_3\mathcal{R}p_1)\mathcal{R}(p_1\mathcal{R}p_0)$ generate a balanced binary parse tree where the maximum depth of recursion is $O(\log\ell)$. However if the nesting is only from one side, such as formulas similar to $p_3\mathcal{R}(p_2\mathcal{R}(p1\mathcal{R}p_0))$, then the maximum depth of recursion is $O(\ell)$. Thus we focus on the formula

$$\varphi = p_\ell\mathcal{R}_{[a_\ell,b_\ell]}(p_{\ell-1}\mathcal{R}_{[a_{\ell-1},b_{\ell-1}]}...\mathcal{R}_{[a_3,b_3]}(p_2\mathcal{R}_{[a_2,b_2]}(p_1\mathcal{R}_{[a_1,b_1]}p_0))...)$$

where there are ℓ logical connectives $\mathcal{R}$ and $n = \ell+1$ propositional variables.

We derive the complexity of $S(\texttt{reg}(\varphi))$ by defining the sequence $\{s_k\}_{k=1}^{\ell}$ recursively as follows, $s_1 := S(\texttt{reg}(p_1\mathcal{R}_{[a_1,b_1]}p_0))$ and $s_{k+1} := \mathcal{C}_R S(\texttt{reg}(p_{k+1}))(s_k)^{\delta}$, for each $1\leqslant k<\ell$. The recurrence relation captures an extra nesting of the $\mathcal{R}$ operator, based on the complexity of $\mathcal{R}$ defined above. We calculate $S(p_m) = n = \ell+1$ for all m such that $0\leqslant m\leqslant\ell$, thus $s_1 = \mathcal{C}_R(\ell+1)^{\delta+1}$ and $s_{k+1} = \mathcal{C}_R(\ell+1)(s_k)^{\delta}$.

The explicit formula is given by $s_k = \mathcal{C}_R^{\sum_{i=0}^{k-1}\delta^i}\cdot(\ell+1)^{\sum_{i=0}^{k}\delta^i}$. It is easy to check that the base case $k=1$ holds, and we prove the claim by induction:

$$\begin{aligned}
S_{k+1} &= \mathcal{C}_R\left(\mathcal{C}_R^{\sum_{i=0}^{k-1}\delta^i}\cdot(\ell+1)^{\sum_{i=0}^{k}\delta^i}\right)^{\delta}\cdot(\ell+1) \\
&= \mathcal{C}_R^{1+\sum_{i=0}^{k-1}\delta^{i+1}}\cdot(\ell+1)^{1+\sum_{i=0}^{k}\delta^{i+1}} \\
&= \mathcal{C}_R^{\sum_{i=0}^{k}\delta^i}\cdot(\ell+1)^{\sum_{i=0}^{k+1}\delta^i}.
\end{aligned}$$

Thus we have that $S(\texttt{reg}(\varphi)) = \mathcal{C}_R^{\sum_{i=0}^{\ell-1}\delta^i}\cdot(\ell+1)^{\sum_{i=0}^{\ell}\delta^i} = O(\mathcal{C}_R^{\delta^{\ell}}\cdot(\ell+1)^{\delta^{\ell+1}})$.

Through a similar analysis, we have found that the time complexities of all of the above functions is unsurprisingly the same as their space complexities.

Theorem 6 (***Time Complexity***)**.** *Given a well-formed MLTL formula φ,* $\texttt{reg}(\varphi)$ *has worst-case time complexity that is $O(\mathcal{C}_R^{\delta^\ell} \cdot (\ell+1)^{\delta^{\ell+1}})$, where ℓ is the number of logical connectives in $\{\wedge, \vee, \mathcal{F}, \mathcal{G}, \mathcal{R}, \mathcal{U}\}$ that occurr in φ.*

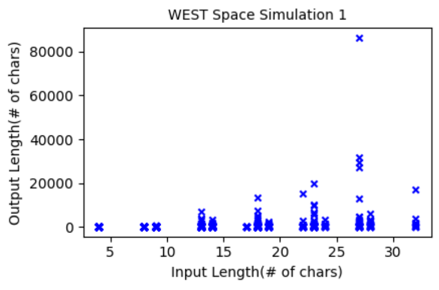
If in the worst case no simplification occurs in any call of `set_intersect`, space complexity remains unchanged, but simplifying a set of regular expressions is cubic in input size. In practice, however, both time and space complexities are much more optimistic than worst-case estimates.

4.3 Experimental Benchmarking

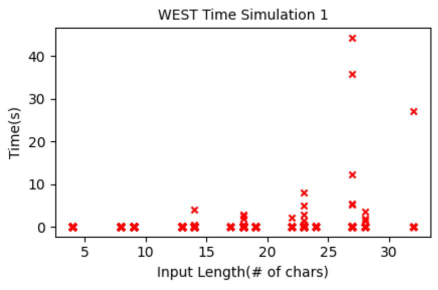
We accompany theoretical space and time complexity with experimental evaluation of these complexities using randomly-generated MLTL formulas. What we observed from the simulations is that the worst-case complexity, both for space and time, is relatively rare, and that otherwise the program has good complexity. WEST ran in under 10 seconds for nearly all the inputted random formulas. The number of characters outputted was typically under 5000, and often less. This is approximately the length of a single paragraph. We also observe that these worst cases are extreme outliers and that in nearly every case, are examples of nested binary temporal operators. As seen in [14], [16], [2], and [13], nested binary temporal operators do not appear in any specifications, and thus are unlikely to appear both in the literature and in practical applications.

We ran these experiments on an Intel(R) Xeon(R) Gold 6140 CPU @ 2.30GHz with 376 GB RAM. For each simulation, 1000 MLTL formulas were randomly generated using the parameters `delta`, `interval_max`, number of propositional variables, and number of iterations. Here, `delta` is the maximum length we allow for any interval, `interval_max` is the largest allowed upper bound for any interval, and Number of iterations is the level of nesting in the generated formulas. For example, $\mathcal{G}_{[0,2]}p_0$ is a formula generated with one iteration, while $\mathcal{G}_{[0,2]}(p_0 \wedge p_1)$ is a formula generated with two iterations. We measure the number of characters in the output versus time in seconds taken to run the program. For the pseudocode of the program that generated the random formulas, we refer the reader to check the WEST Appendix[9].

Simulation 1 For the first simulation, we consider 2 iterations, 5 propositional variables, $\texttt{delta} = 10$, and $\texttt{interval\_max} = 10$. We obtain plots 2a and 2b.



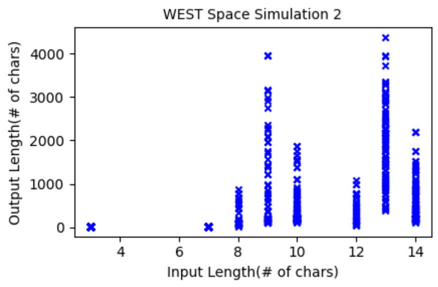
(a)



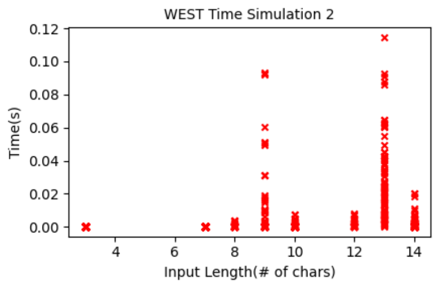
(b)

Fig. 2: $(p_0 = p_1)\mathcal{U}_{[2,9]}(p_1\mathcal{U}_{[7,9]}p_3)$ is an outlier in both. $(p_4 \rightarrow p_2)\mathcal{R}_{[1,8]}(p_3\mathcal{U}_{[3,4]}p_0)$ and $(p_3\mathcal{R}_{[2,7]}p_4)\mathcal{R}_{[1,9]}(p_4\mathcal{R}_{[4,9]}p_3)$ are outliers only in b.

Simulation 2 For the second simulation, we consider 1 iteration, 10 propositional variables, `delta` = 20, `interval_max` = 20. We obtain plots 3a and 3b.



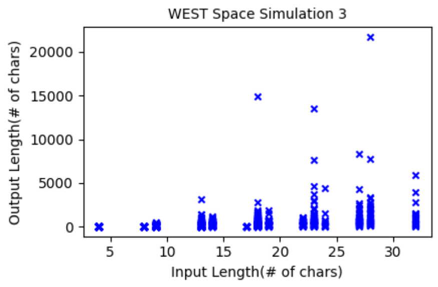
(a)



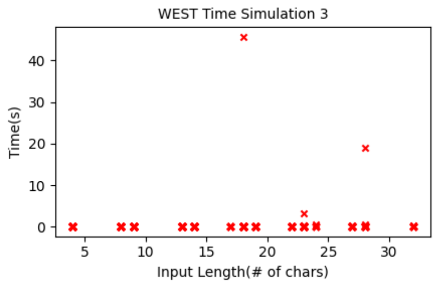
(b)

Fig. 3: We observe no outliers. The zero second runtimes are observed for MLTL formulas that consist of a single propositional variable or its negation.

Simulation 3 For the third simulation, we consider 2 iterations, 10 propositional variables, `delta` = 5, and `interval_max` = 10. We obtain plots 4a and 4b.



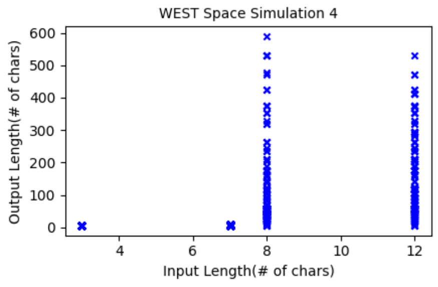
(a)



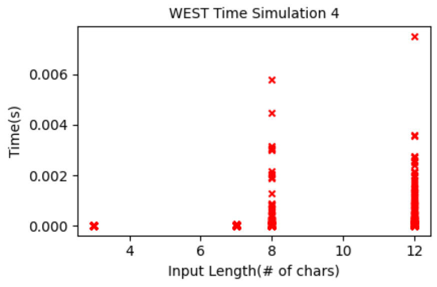
(b)

Fig. 4: $(p_7\mathcal{U}_{[5,7]}p_9)\mathcal{U}_{[3,7]}(\mathcal{F}_{[1,4]}p_4)$ and $\mathcal{G}_{[3,7]}(p_9\mathcal{U}_{[0,4]}p_0)$ are outliers in both. $\mathcal{G}_{[3,7]}(p_9\mathcal{U}_{[0,4]}p_0)$ is an outlier in 4a only.

Simulation 4 For the fourth simulation, we consider 1 iteration, 5 propositional variables, `delta` = 10, `interval_max` = 10. We obtain plots 5a and 5b.



(a)



(b)

Fig. 5: We observe no outliers. The zero second runtimes are observed for MLTL formulas that consist of a single propositional variable or its negation.

We conclude from the simulations that for most practical purposes where nesting of binary temporal operators rarely occur, the WEST algorithm demonstrates good space and time complexity.

5 Correctness of WEST Tool Implementation

We accompany our proof of algorithmic correctness with a rigorous evaluation of implementation correctness, showing that our WEST tool correctly implements our WEST algorithm. Since our proof of algorithmic correctness is manual and our focus is on usability for validation by humans, we utilize more traditional techniques for robust software engineering with testing-based evaluation. The naïve approach is to test all inputs up to a certain size and verify the outputs, but this strategy would generate an unnecessarily large and redundant test suite. For instance, there is little sense in testing all MLTL formulas of the form $p_0\mathcal{U}_{[0,t]}p_1$ for all t such that $0 \leqslant t \leqslant 99$; verification of a few should give sufficient confidence of correctness of the program. Instead, we test our implementation with a test suite that explores all possible sequences of lines of code that are executed (up to a certain depth).

5.1 Intelligent Fuzzing

Traditional black-box fuzzing is defined by Miller [23]: *"If we consider a program to be a complex finite state machine, then our testing strategy can be thought of as a random walk through the state space, searching for undefined states."*

Instead we utilize intelligent fuzzing, an alternate approach that leverages knowledge about program structure to generate valid inputs and increase coverage. Borrowing the words of Miller, our approach to testing the WEST program can be thought of as walking all possible paths up to a certain depth of the state space of our algorithm. We first outline our overall approach to intelligent fuzzing:

1. Construct a directed graph representation of our algorithm. The edges capture control flow of our algorithm, and vertices represent non-branching blocks of code.
2. Construct a test suite that explores all possible paths in the directed graph up to a certain depth. Run the WEST program on the test suite to produce a set of output files.
3. Run a naïve brute force generator of satisfying computations on the test suite and verify that both outputs match for all test cases.

State Diagram Construction We can represent the state space of the WEST algorithm as a directed graph with the edges representing the control flow and vertices representing blocks of contiguous code without branching statements. The core of the WEST program lies in the recursive routine, `reg`, which calls the 8 different subroutines as shown in Fig. 6.

In order to construct the intelligent fuzzing test suite, we make the design choice to abstract away the eight subroutines in the overall state space diagram, despite the fact that they may have different possible execution paths within them. Without this abstraction, attempting to explore all execution paths in this finer graph is infeasible due to the explosion in the number of paths [26], some of which are provably impossible to construct a test input to explore.

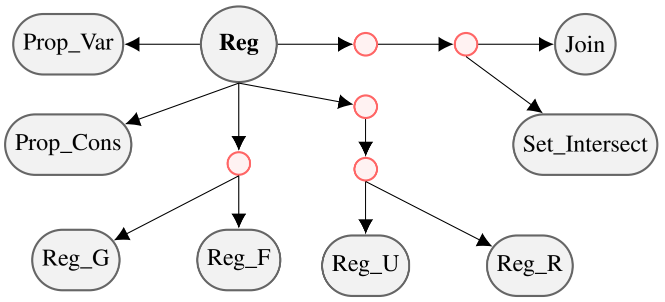


Fig. 6: Abstracted graph of the `reg` main routine. Red nodes signal recursive calls to `reg` on subformulas of the input formula.

Creating the Test Suite To generate our intelligent fuzzing test suite, we first count the number of formulas $\varphi(d)$ to be generated as a function of the exact depth d of recursion desired. For $d = 0$, only the paths leading to `prop_var` and `prop_cons` can be explored, so $\varphi(0) = 2$. For $d \geqslant 1$, we recursively calculate $\varphi(d+1) = 2\varphi(d)+4\varphi(d)^2$; paths to `reg_G` and `reg_F` is counted by the linear term, and paths to `reg_U`, `reg_R`, `set_intersect`, and `join` is counted by the quadratic term. This gives $\varphi(1) = 20$, $\varphi(2) = 1640$, and $\varphi(3) = 10761680$, which tells us that $d = 3$ is computationally infeasible and $d = 1$ does not give us assurance about operators interacting with each other through nesting. Whence we select $d = 2$ as a happy medium. We generate the full test suite in a similar recursive manner. Firstly, the $d = 0$ test suite consists of two formulas: a propositional variable or its negation, and a propositional constant. Then for any $d \geqslant 1$, we iterate through all formulas in the depth $d - 1$ test suite for `reg_G` and `reg_F`, and all pairs of formulas from the $d - 1$ test suite for the remaining four recursive paths. To ensure wider coverage, we randomly generate each of the propositional variables or their negation and propositional constants.

Verifying against Naïve Brute Force A relatively straightforward approach to generating the set of all satisfying computations of an MLTL formula φ over n variables, such that $m = \text{cplen}(\varphi)$, is to iterate over all $2^{m \cdot n}$ possible computations, which counts all possible length $m \cdot n$ bit strings. An interpreter function takes computation π and MLTL formula φ and determines if $\pi \models \varphi$ based purely on MLTL semantics. Our test program translates every first-order quantifier into a loop; then checking for satisfying conditions of the suffix of a computation naturally lends itself to recursion. The full implementation details are available in the WEST [Github][5]. On an Intel(R) Core(TM) i7-4770S CPU at 3.10GHz with 32gb RAM, the brute force program took nearly nine hours to execute the depth two test suite of 1640 formulas. For this test suite, we fixed the number of propositional variables at $n = 4$ and the largest computation length was $m = 5$, from formulas with doubly-nested temporal operators.

In comparison, the WEST program executed the same test suite in under thirty minutes on the same machine. Note that the brute force program outputs only computations of zeros and ones, and thus comparing the outputs of the WEST program requires expanding out the "S" characters in the regular expressions. It is important to state that although the full test suite matches between both implementations, absolute correctness on all inputs is not guaranteed for either program. However, the successful execution of the test suite gives us a much higher confidence in correctness of both the WEST program and the brute force program.

6 Regular Expression Simplification Theorem (REST)

As a final result, we provide a regular expression simplification theorem. This theorem describes the form of a set of MLTL regular expressions that simplify to all "S" characters. This theorem may help users identify tautologies, as the WEST program does not always output a string of all "S" characters when a formula holds true for every computation. We first define some vocabulary. We call an *arbitrary computation* any regular expression composed entirely of "S" characters and commas. For the purposes of the following theorem, we remove all commas from computations. We say a "0" or a "1" in a regular expression is a *fixed truth value*. We overload the definition of a matrix and say that a *matrix* is a representation of a union of regular expressions, where each row is a regular expression. This aids significantly in the description and proof of the theorem. Note that usual matrix algebra is not relevant to this definition.

Theorem 7 (***Regular Expression Simplification Theorem***)**.** *Let M be an $(n+1) \times n$ matrix, where each of the $n+1$ rows represents a regular expression of length n with commas stripped. If each column has one "1", one "0", and $n-1$ "S" characters, then the union of this set of regular expressions can be simplified to S^n, the arbitrary computation of length n.*

The proof of REST follows from induction on the size of M and the pigeon hole principle. See the WEST Appendix[9] for details of the proof.

This theorem gives us a sufficient but not a necessary condition for simplification to the arbitrary computation. One such example is the regular expression $(101) \vee (S1S) \vee (1S0) \vee (0SS)$, which fails the hypothesis of the simplification theorem but is still equivalent to SSS.

6.1 Theoretical Analysis of REST

We present an algorithm based on Theorem 7 for simplifying disjunctions of regular expressions and provide theoretical analysis, as well as experimental benchmarking. We determine that REST runs exponentially with respect to the length of the inputted regular expressions, both in the worst-case and average case. However, REST does not apply for many MLTL formulas, and WEST already demonstrates good time and space complexity in the average case without REST (see section 4.3). Thus, the algorithmic complexity of REST is not of practical concern.

Theorem 8. *On input vector of regular expressions v of m strings each of length n, REST has a worst-case runtime of $O(n^2 2^m)$.*

Proof. We first analyze the statements in the innermost loop. In line 4, the construction of *diff_cols* can be done in $O(nr)$ time, by iteratively scanning the columns of w. In line 7, checking if w' satisfies the conditions of Theorem 7 is done in $O(r^2)$ time, by keeping a count of the number of $0, 1, S$ in each column. Thus, the innermost portion

Algorithm 5 Regular Expression Simplification Algorithm (REST)

Inputs: vector v of m regular expressions each of length n
Output: simplified set of equivalent regular expressions using Theorem 7

```
1: procedure REST(set $v$)
2:     for $r \in [3, \min(m, n+1)]$ do
3:         for all vectors of regular expression $w \subseteq v$ s.t. $|v| = r$ do
4:             diff_cols ← indices of all columns of $w$ that are not uniformly the same character
5:             if $|\text{diff\_cols}| = r - 1$ then
6:                 $w' \leftarrow w$ containing only columns $\in$ diff_cols
7:                 if $w'$ satisfies Theorem 7 then
8:                     In $w$, replace all columns $\in$ diff_cols with $s$
9:                     $v \leftarrow$ remove_duplicates($v$)
10:     return $v$
```

of the loop has runtime $O(nr + r^2) = O(n^2)$. The total runtime is bounded as follows:

$$\text{runtime}(\text{REST}(v)) = \sum_{r=3}^{min(m,n+1)} \binom{m}{r} O(n^2) = O(n^2 2^m).$$

6.2 Experimental Benchmarking of REST

We provide an experimental evaluation of the runtime of REST using randomly-generated sets of regular expressions satisfying the conditions of REST. Unfortunately, results suggest that the average case time complexity is of the worst case.

We generated 100 sets of regular expressions satisfying the REST conditions, with n between 10 and 25. We measured the amount of time in seconds taken to run the program. We ran these experiments on an Intel(R) Xeon(R) Gold 6140 CPU @ 2.30GHz with 376 GB RAM, taking over one hour. We conclude that REST is not advisable to use as a part of the WEST program because it is often too computationally expensive.

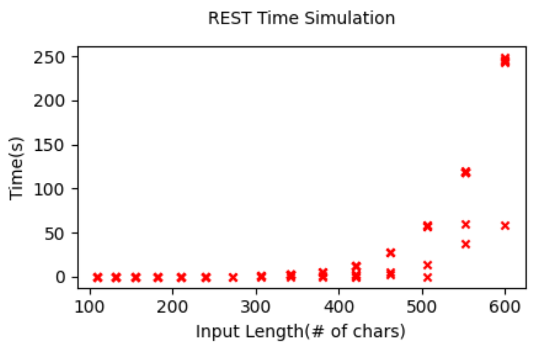


Fig. 7: For most inputs, REST's runtime is exponential with respect to the input length.

7 Using WEST: An Example

For this section, we include a Video tutorial[10] that demonstrates the use the WEST GUI program to explore specifications. The video uses the formula $(p_0 \wedge \mathcal{G}_{[0,3]}p_1) \rightarrow p_2$ from [16] as an example. The WEST program interface can aid a user in the process of MLTL formula validation by allowing them to explore the behavior of the formula and its subformulas.

The user can toggle the value of propositional variables at individual time steps to explore if the resulting computation satisfies the given formula. Additionally, the tool can randomly generate a satisfying computation that matches a specific regular expression, randomly generate an unsatisfying computation, and perform backbone analysis. All of these functionalities are immensely useful for allowing a user to validate that the formula they have written means what they think it means.

8 Closing Remarks

The primary goal of this work is to visually represent MLTL formulas to aid in debugging of MLTL specifications in industrial domains. We have accomplished this with our regular expressions framework, which captures many structural patterns of satisfying computations for a given MLTL formula. The tool itself has demonstratively reasonable runtime for most inputs, and the correctness of outputs has been verified to a high degree of confidence through intelligent fuzzing.

8.1 Future Work

The WEST algorithm and the release of our open-source tool open a multitude of different research directions.

Similar Analysis of LTL ω-regular expressions match only infinite words and can be used to describe satisfying computations of LTL formulas. Our work on validation for MLTL formulas lays the groundwork for future work on validation of the finite-trace logic LTLf [9] as well as LTL formulas. For the infinite-trace semantics of LTL, the particular difficulties revolve around the Kleene star operation, which behaves poorly due to computing infinite unions and intersections of regular expressions.

Regular Expression Simplification Theorem 7 addresses a non-trivial situation in which a set of regular expressions may be simplified. A natural question to ask is what is the minimum number of regular expressions needed to represent a language of computations. However, such a minimal representation is not unique. For instance, $\{S1, 1S\} = \{S1, 10\} = \{01, 10, 11\}$. Other schemes may be needed to simplify any arbitrary union of regular expressions to a minimal representation.

Fuzzing General Recursive Algorithms A tool to systematically convert a recursive algorithm into a directed graph representation of the state space would be a helpful aid for generating test suites. Additional care should be put into allowing for varying levels of abstractions of execution due to concerns of the path explosion problem. We note that such avenues for code-level verification will still be necessary even upon the completion of potential future work avenues like synthesizing the core implementation of the WEST algorithm from an interactive theorem prover. This is because the goal

[10] https://youtu.be/HoBJwdCq42c

of explainability to humans will require at least some manual code authorship for the foreseeable future.

Code Synthesis from MLTL Specification The truth tables generated by the WEST tool can now serve as input to a recently-published toolchain [3]. This newly-enabled workflow would produce encodings of the represented MLTL behavior for the interactive theorem prover ACL2, including automatically generating related properties of general interest such as unambiguousness [4,15,6]. Next, a synthesis pipeline consisting of a verified program transformation suite [18] along with a proof-generating C code generator [5] (both built on ACL2) generates verified software implementing the behaviors originally described in MLTL. By providing a new front-end for this tool chain, we have now enabled a path to generating provably correct software from validated MLTL formulas describing the desired behaviors of a system. Considering the rising popularity of MLTL for describing such behaviors, we expect this to be a rewarding avenue for future exploration.

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Appendix

I Minimum Computation Length

Theorem 1 (***Minimum Computation Length of Until and Release***)**.** *Let* $0 \leqslant a \leqslant b \in \mathbb{N}$ *and let* φ, ψ *be well-formed MLTL formulas in NNF. The minimum computation length of* Until *and* Release *is given by* $\text{cplen}(\varphi \mathcal{U}_{[a,b]} \psi) = \text{cplen}(\varphi \mathcal{R}_{[a,b]} \psi) = b + \max(\text{cplen}(\varphi) - 1, \text{cplen}(\psi))$.

Proof. Recall that $\text{reg}(\varphi\, \mathcal{U}_{[a,b]} \psi) = \bigvee_{i=a}^{b} \text{reg}\left(\mathcal{G}_{[a,i-1]}\varphi \wedge \mathcal{G}_{[i,i]}\psi\right)$. Thus

$$\begin{aligned}
\text{cplen}(\varphi \mathcal{U}_{[a,b]} \psi) &= \max_{a \leqslant i \leqslant b} \left(\text{cplen}(\mathcal{G}_{[a,i-1]}\varphi \wedge \mathcal{G}_{[i,i]}\psi)\right) \\
&= \max_{a \leqslant i \leqslant b} \left(\max(i - 1 + \text{cplen}(\varphi), i + \text{cplen}(\psi))\right) \\
&= \max_{a \leqslant i \leqslant b} \left(i + \max(\text{cplen}(\varphi) - 1, \text{cplen}(\psi))\right) \\
&= b + \max(\text{cplen}(\varphi) - 1, \text{cplen}(\psi)).
\end{aligned}$$

Now recall that $\text{reg}(\varphi \mathcal{R}_{[a,b]} \psi) = \text{reg}\left(\mathcal{G}_{[a,b]}\psi\right) \vee \bigvee_{i=a}^{b-1} \text{reg}\left(\mathcal{G}_{[a,i]}\psi \wedge \mathcal{G}_{[i,i]}\varphi\right)$. Thus

$$\begin{aligned}
\text{cplen}(\varphi \mathcal{R}_{[a,b]} \psi) &= \max\left(\text{cplen}(\mathcal{G}_{[a,b]}\psi), \text{cplen}\left(\bigvee_{i=a}^{b-1} \text{reg}\left(\mathcal{G}_{[a,i]}\psi \wedge \mathcal{G}_{[i,i]}\varphi\right)\right)\right) \\
&= \max\left(b + \text{cplen}(\psi), \max_{a \leqslant i \leqslant b-1} \left(\mathcal{G}_{[a,i]}\psi \wedge \mathcal{G}_{[i,i]}\varphi\right)\right) \\
&= \max\left(b + \text{cplen}(\psi), \max_{a \leqslant i \leqslant b-1} \left(\max(i + \text{cplen}(\psi), i + \text{cplen}(\varphi))\right)\right) \\
&= \max\left(b + \text{cplen}(\psi), \max(b - 1 + \text{cplen}(\varphi), b - 1 + \text{cplen}(\psi))\right) \\
&= \max(b + \text{cplen}(\psi), b - 1 + \text{cplen}(\varphi), b - 1 + \text{cplen}(\psi)) \\
&= b + \max(\text{cplen}(\varphi) - 1, \text{cplen}(\psi)).
\end{aligned}$$

II Soundness and Completeness

Theorem 2 (***Soundness and Completeness***)**.** *For any well formed MLTL formula* φ *in negation normal form, a computation* π *with* $|\pi| = cplen(\varphi)$ *satisfies* φ *and if and only if* $\pi \in \mathscr{L}(\text{reg}(\varphi))$.

Proof. We proceed by induction on the length of the formula.
Base Cases.
reg($\top$)
Any computation π satisfies $\top$, and $\mathscr{L}(\text{reg}(\top))$ is the set of all computations of one time step. Padding conventions extends this to the set of all computations of any positive number of time steps, so we are done.
reg($\bot$)
There are no computations that satisfy $\bot$ and $\mathscr{L}(\text{reg}(\bot))$ is the empty string, so we are done.
reg(p_k)

For $0 \leqslant k \leqslant n-1$, consider the propositional variable p_k. If π satisfies p_k, then p_k evaluates to true at $\pi[0]$. This is equivalent to writing that $\pi[0]$ is of the form

$$S^k 1 S^{n-k-1}$$

, which is precisely reg(p_k).
reg($\neg p_k$)
If π satisfies $\neg p_k$, then $\neg p_k$ evaluates to true at $\pi[0]$. This is equivalent to writing that $\pi[0]$ is of the form

$$S^k 0 S^{n-k-1}$$

, which is precisely reg$(\neg p_k)$.
Inductive Step.
Suppose the theorem holds for MLTL formulas φ and ψ. We now show that it holds for $\varphi \wedge \psi$, $\varphi \vee \psi$, $\mathcal{F}_{[a,b]}\varphi$, $\mathcal{G}_{[a,b]}\varphi$, $\varphi\,\mathcal{U}_{[a,b]}\psi$, and $\varphi\mathcal{R}_{[a,b]}\psi$.
reg($\varphi \wedge \psi$)
We know that $\pi \models \varphi \wedge \psi$ iff $\pi \models \varphi$ and $\pi \models \psi$.
By the inductive hypothesis, $\pi \models \varphi$ iff $\pi \in \mathscr{L}(\text{reg}(\varphi))$ and
$\pi \models \psi$ iff $\pi \in \mathscr{L}(\text{reg}(\psi))$, so

$$\pi \models \varphi \wedge \psi \text{ iff } \pi \in (\mathscr{L}(\text{reg}(\varphi)) \wedge \mathscr{L}(\text{reg}(\psi))) = \mathscr{L}(\text{reg}(\varphi \wedge \psi)).$$

reg($\varphi \vee \psi$)
We know that $\pi \models \varphi \vee \psi$ iff $\pi \models \varphi$ or $\pi \models \psi$.
By the inductive hypothesis, $\pi \models \varphi$ iff $\pi \in \mathscr{L}(\text{reg}(\varphi))$ and
$\pi \models \psi$ iff $\pi \in \mathscr{L}(\text{reg}(\psi))$, so

$$\pi \models \varphi \vee \psi \text{ iff } \pi \in (\mathscr{L}(\text{reg}(\varphi)) \vee \mathscr{L}(\text{reg}(\psi))) = \mathscr{L}(\text{reg}(\varphi \vee \psi)).$$

reg($\mathcal{F}_{[a,b]}\varphi$)
We know that $\pi \models \mathcal{F}_{[a,b]}\varphi$ iff $|\pi| > a$ and $\exists i \in [a,b]$ such that $\pi_i \models \varphi$.
If $|\pi| = \text{cplen}(\mathcal{F}_{[a,b]}\varphi)$, $|\pi| > a$. Likewise, $\pi \in \mathscr{L}\left(\text{reg}\left(\mathcal{F}_{[a,b]}\varphi\right)\right)$ implies $|\pi| = \text{cplen}(\mathcal{F}_{[a,b]}\varphi)$, so the length condition is satisfied.
By the inductive hypothesis, $\pi_i \models \varphi$ iff $\pi_i \in \mathscr{L}(\text{reg}(\varphi))$, so $\pi \in \mathscr{L}((S^n,)^i\text{reg}(\varphi))$ for some $i \in [a,b]$. Equivalently,

$$\pi \in \mathscr{L}\left(\bigvee_{i=a}^{b}(S^n,)^i\text{reg}(\varphi)(,S^n)^{b-i}\right) = \mathscr{L}(\text{reg}(\mathcal{F}_{[a,b]}\varphi)).$$

reg($\mathcal{G}_{[a,b]}\varphi$)
We know that $\pi \models \mathcal{G}_{[a,b]}\varphi$ iff $|\pi| \leqslant a$ or $\forall i \in [a,b]\ \pi_i \models \varphi$.
Since $|\pi| = \text{cplen}(\mathcal{G}_{[a,b]}\varphi)$ and $\text{cplen}(\mathcal{G}_{[a,b]}\varphi) > a$, the first option for satisfying the formula never occurs.
By the inductive hypothesis, $\pi_i \models \varphi$ iff $\pi_i \in \mathscr{L}(\text{reg}(\varphi))$, so $\pi \in \mathscr{L}((S^n,)^i\text{reg}(\varphi))$ for all $i \in [a,b]$. Equivalently,

$$\pi \in \mathscr{L}\left(\bigwedge_{i=a}^{b}(S^n,)^i\text{reg}(\varphi)(,S^n)^{b-i}\right) = \mathscr{L}(\text{reg}(\mathcal{F}_{[a,b]}\varphi)).$$

reg($\varphi\,\mathcal{U}_{[a,b]}\psi$)
We have that

$$\pi \vDash \varphi\,\mathcal{U}_{[a,b]}\psi \text{ iff } |\pi| > a \text{ and } \exists i \in [a,b] \text{ such that } (\pi_i \vDash \psi \text{ and } \forall a \leqslant j < i, \pi_j \vDash \varphi).$$

As argued in the Finally case, the length condition is satisfied.
By the inductive hypothesis, $\pi_i \vDash \psi$ iff $\pi_i \in \mathscr{L}(\text{reg}(\psi))$, or equivalently, $\pi \in \mathscr{L}(\text{reg}(\mathcal{G}_{[i,i]}\psi))$. Also, $\pi_j \vDash \varphi$ if and only if $\pi_j \in \mathscr{L}(\text{reg}(\varphi))$.
By the argument used in the Global case, we see that $\forall a \leqslant j < i,\ \pi_j \vDash \varphi$ is equivalent to $\pi \in \mathscr{L}(\text{reg}(\mathcal{G}_{[a,i-1]}\varphi))$.
Thus $\pi \vDash \varphi\,\mathcal{U}_{[a,b]}\psi$ iff $\pi \in \mathscr{L}((\text{reg}(\mathcal{G}_{[i,i]}\psi) \wedge \text{reg}(\mathcal{G}_{[a,i-1]}\varphi)))$ for some $i \in [a,b]$, or equivalently,

$$\pi \in \mathscr{L}\left(\bigvee_{i=a}^{b} \text{reg}\left(\mathcal{G}_{[a,i-1]}\varphi \wedge \mathcal{G}_{[i,i]}\psi\right)\right) = \mathscr{L}\left(\text{reg}(\varphi\,\mathcal{U}_{[a,b]}\psi)\right).$$

reg($\varphi\mathcal{R}_{[a,b]}\psi$)
We have that

$$\begin{aligned}&\pi \vDash \varphi\mathcal{R}_{[a,b]}\psi \text{ iff } |\pi| \leqslant a \text{ or } \forall i \in [a,b]\ \pi_i \vDash \psi \text{ or}\\ &\exists j \in [a,b] \text{ such that } (\pi_j \vDash \varphi \text{ and } \forall a \leqslant k \leqslant j, \pi_k \vDash \psi).\end{aligned}$$

As argued in the Global case, the first option for satisfying the formula never occurs.
By the Global case, the statement $\forall i \in [a,b]\ \pi_i \vDash \psi$ is equivalent to $\pi \in \mathscr{L}(\text{reg}(\mathcal{G}_{[a,b]}\psi))$ and, by the Finally case. the statement $\exists j \in [a,b]$ such that ($\pi_j \vDash \varphi$ and $\forall a \leqslant k \leqslant j$, $\pi_k \vDash \psi$) is equivalent to $\pi \in \mathscr{L}\left(\bigvee_{i=a}^{b} \text{reg}\left(\mathcal{G}_{[a,i]}\psi \wedge \mathcal{G}_{[i,i]}\varphi\right)\right)$. Hence

$$\begin{aligned}\pi \vDash \varphi\mathcal{R}_{[a,b]}\psi \text{ iff } \pi \in \mathscr{L}&\left(\text{reg}(\mathcal{G}_{[a,b]}\psi) \vee \bigvee_{i=a}^{b} \text{reg}\left(\mathcal{G}_{[a,i]}\psi \wedge \mathcal{G}_{[i,i]}\varphi\right)\right)\\ = \mathscr{L}&\left(\text{reg}(\mathcal{G}_{[a,b]}\psi) \vee \bigvee_{i=a}^{b-1} \text{reg}\left(\mathcal{G}_{[a,i]}\psi \wedge \mathcal{G}_{[i,i]}\varphi\right)\right)\\ = \mathscr{L}&\left(\text{reg}(\varphi\mathcal{R}_{[a,b]}\psi)\right).\end{aligned}$$

This completes the inductive step, and thus the proof. Since this proof addresses all possible MLTL formulas in negation normal form, it shows completeness along with soundness.

III Nested Until and Release Rewriting Theorem

Theorem 3 (***Nested Until and Release Rewriting Theorem***). *Any MLTL formula using the Until or Release operator can be rewritten with right-nested subformulas. Let $\mathcal{B} = \mathcal{R}$ or $\mathcal{U}$. Let $a, b, c \in \mathbb{Z}_{\geqslant 0}$, $a \leqslant b$, and φ, ψ be well-formed MLTL formulas in NNF. Then, $\varphi\ \mathcal{B}_{[a,b+c]}\psi \equiv \varphi\ \mathcal{B}_{[a,b]}(\varphi\ \mathcal{B}_{[0,c]}\psi)$. That is, $\varphi\ \mathcal{U}_{[a,b+c]}\psi \equiv \varphi\ \mathcal{U}_{[a,b]}(\varphi\ \mathcal{U}_{[0,c]}\psi)$ and $\varphi\ \mathcal{R}_{[a,b+c]}\psi \equiv \varphi\ \mathcal{R}_{[a,b]}(\varphi\ \mathcal{R}_{[0,c]}\psi)$.*

The proof of Theorem 3 appears below.

Proof. **Case 1:** $\mathcal{B} = \mathcal{U}$
Let $\gamma = \varphi\,\mathcal{U}_{[0,c]}\psi$. Then, $\varphi\,\mathcal{U}_{[a,b]}(\varphi\,\mathcal{U}_{[0,c]}\psi) = \varphi\,\mathcal{U}_{[a,b]}\gamma$ and
$\text{reg}\,(\gamma) = \bigvee_{j=0}^{c} \text{reg}\left(\mathcal{G}_{[0,j-1]}\varphi \wedge \mathcal{G}_{[j,j]}\psi\right)$. Thus

$$\begin{aligned}
\text{reg}\left(\varphi\,\mathcal{U}_{[a,b]}\gamma\right) &= \bigvee_{i=a}^{b} \text{reg}\left(\mathcal{G}_{[a,i-1]}\varphi \wedge \mathcal{G}_{[i,i]}\text{reg}\,(\gamma)\right) \\
&= \bigvee_{i=a}^{b} \text{reg}\left(\mathcal{G}_{[a,i-1]}\varphi \wedge \mathcal{G}_{[i,i]}\left(\bigvee_{j=0}^{c} \text{reg}\left(\mathcal{G}_{[0,j-1]}\varphi\right) \wedge \text{reg}\left(\mathcal{G}_{[j,j]}\psi\right)\right)\right).
\end{aligned}$$

$\mathcal{G}_{[i,i]}$ distributes over $\wedge$ and $\vee$, so

$$\begin{aligned}
\text{reg}\left(\varphi\,\mathcal{U}_{[a,b]}\gamma\right) &= \bigvee_{i=a}^{b} \text{reg}\left(\mathcal{G}_{[a,i-1]}\varphi \wedge \bigvee_{j=0}^{c} \mathcal{G}_{[i,i]}\left(\text{reg}\left(\mathcal{G}_{[0,j-1]}\varphi\right) \wedge \text{reg}\left(\mathcal{G}_{[j,j]}\psi\right)\right)\right) \\
&= \bigvee_{i=a}^{b} \text{reg}\left(\mathcal{G}_{[a,i-1]}\varphi \wedge \left(\bigvee_{j=0}^{c} \mathcal{G}_{[i,i]}\text{reg}\left(\mathcal{G}_{[0,j-1]}\varphi\right) \wedge \mathcal{G}_{[i,i]}\text{reg}\left(\mathcal{G}_{[j,j]}\psi\right)\right)\right).
\end{aligned}$$

Since $\mathcal{G}_{[t_1,t_1]}\mathcal{G}_{[t_2,t_3]}\varphi \equiv \mathcal{G}_{[t_1+t_2,t_1+t_3]}\varphi$, we have

$$\begin{aligned}
\text{reg}\left(\varphi\,\mathcal{U}_{[a,b]}\gamma\right) &= \bigvee_{i=a}^{b} \text{reg}\left(\mathcal{G}_{[a,i-1]}\varphi \wedge \left(\bigvee_{j=0}^{c} \text{reg}\left(\mathcal{G}_{[i,i+j-1]}\varphi\right) \wedge \text{reg}\left(\mathcal{G}_{[i+j,i+j]}\psi\right)\right)\right) \\
&= \bigvee_{i=a}^{b}\bigvee_{j=0}^{c} \text{reg}\left(\mathcal{G}_{[a,i-1]}\varphi\right) \wedge \text{reg}\left(\mathcal{G}_{[i,i+j-1]}\varphi\right) \wedge \text{reg}\left(\mathcal{G}_{[i+j,i+j]}\psi\right).
\end{aligned}$$

Since $\mathcal{G}_{[t_1,t_2-1]}\varphi \wedge \mathcal{G}_{[t_2,t_3]}\varphi \equiv \mathcal{G}_{[t_1,t_3]}\varphi$, we have

$$\begin{aligned}
\text{reg}\left(\varphi\,\mathcal{U}_{[a,b]}\gamma\right) &= \bigvee_{i=a}^{b}\bigvee_{j=0}^{c} \text{reg}\left(\mathcal{G}_{[a,i+j-1]}\varphi\right) \wedge \text{reg}\left(\mathcal{G}_{[i+j,i+j]}\psi\right) \\
&= \bigvee_{i+j=a}^{b+c} \text{reg}\left(\mathcal{G}_{[a,i+j-1]}\varphi\right) \wedge \text{reg}\left(\mathcal{G}_{[i+j,i+j]}\psi\right).
\end{aligned}$$

Finally, let $k = i + j$. Thus

$$\begin{aligned}
\text{reg}\left(\varphi\,\mathcal{U}_{[a,b]}\gamma\right) &= \bigvee_{k=a}^{b+c} \text{reg}\left(\mathcal{G}_{[a,k-1]}\varphi\right) \wedge \text{reg}\left(\mathcal{G}_{[k,k]}\psi\right) \\
&= \text{reg}\left(\varphi\,\mathcal{U}_{[a,b+c]}\psi\right).
\end{aligned}$$

This shows that $\varphi\,\mathcal{U}_{[a,b]}(\varphi\,\mathcal{U}_{[0,c]}\psi) \equiv \varphi\,\mathcal{U}_{[a,b+c]}\psi$.
Case 2: $\mathcal{B} = \mathcal{R}$
We begin by rewriting the regex for Release in an equivalent form to better mirror the

structure of the Until case. The Release operator is the dual of the Until operator. Thus,

$$\begin{aligned}\operatorname{reg}\left(\varphi\,\mathcal{R}_{[a,b]}\psi\right) &= \operatorname{reg}\left(\neg\left(\neg\varphi\,\mathcal{U}_{[a,b]}\neg\psi\right)\right)\\ &= \neg\left(\bigvee_{i=a}^{b}\operatorname{reg}\left(\mathcal{G}_{[a,i-1]}\neg\varphi \wedge \mathcal{G}_{[i,i]}\neg\psi\right)\right).\end{aligned}$$

Global is the dual of Finally, so

$$\begin{aligned}\operatorname{reg}\left(\varphi\,\mathcal{R}_{[a,b]}\psi\right) &= \neg\left(\bigvee_{i=a}^{b}\operatorname{reg}\left(\neg\mathcal{F}_{[a,i-1]}\varphi \wedge \neg\mathcal{F}_{[i,i]}\psi\right)\right)\\ &= \neg\left(\bigvee_{i=a}^{b}\operatorname{reg}\left(\neg\left(\mathcal{F}_{[a,i-1]}\varphi \vee \mathcal{F}_{[i,i]}\psi\right)\right)\right)\\ &= \neg\neg\left(\bigwedge_{i=a}^{b}\operatorname{reg}\left(\mathcal{F}_{[a,i-1]}\varphi \vee \mathcal{F}_{[i,i]}\psi\right)\right)\\ &= \bigwedge_{i=a}^{b}\operatorname{reg}\left(\mathcal{F}_{[a,i-1]}\varphi \vee \mathcal{F}_{[i,i]}\psi\right).\end{aligned}$$

This completes the rewriting of the regex for Release. Now let $\gamma = \varphi\,\mathcal{R}_{[0,c]}\psi$. Then $\varphi\,\mathcal{R}_{[a,b]}(\varphi\,\mathcal{R}_{[0,c]}\psi) = \varphi\,\mathcal{R}_{[a,b]}\gamma$ and $\operatorname{reg}(\gamma) = \bigwedge_{j=0}^{c}\operatorname{reg}\left(\mathcal{F}_{[0,j-1]}\varphi \vee \mathcal{F}_{[j,j]}\psi\right)$. Thus

$$\begin{aligned}\operatorname{reg}\left(\varphi\,\mathcal{R}_{[a,b]}\gamma\right) &= \bigwedge_{i=a}^{b}\operatorname{reg}\left(\mathcal{F}_{[a,i-1]}\varphi \vee \mathcal{F}_{[i,i]}\operatorname{reg}(\gamma)\right)\\ &= \bigwedge_{i=a}^{b}\operatorname{reg}\left(\mathcal{F}_{[a,i-1]}\varphi \vee \mathcal{F}_{[i,i]}\left(\bigwedge_{j=0}^{c}\operatorname{reg}\left(\mathcal{F}_{[0,j-1]}\varphi\right) \vee \operatorname{reg}\left(\mathcal{F}_{[j,j]}\psi\right)\right)\right).\end{aligned}$$

$\mathcal{F}_{[i,i]} \equiv \mathcal{G}_{[i,i]}$, so this operation distributes over $\wedge$ and $\vee$. Thus

$$\begin{aligned}\operatorname{reg}\left(\varphi\,\mathcal{R}_{[a,b]}\gamma\right) &= \bigwedge_{i=a}^{b}\operatorname{reg}\left(\mathcal{F}_{[a,i-1]}\varphi \vee \bigwedge_{j=0}^{c}\mathcal{F}_{[i,i]}\left(\operatorname{reg}\left(\mathcal{F}_{[0,j-1]}\varphi\right) \vee \operatorname{reg}\left(\mathcal{F}_{[j,j]}\psi\right)\right)\right)\\ &= \bigwedge_{i=a}^{b}\operatorname{reg}\left(\mathcal{F}_{[a,i-1]}\varphi \vee \left(\bigwedge_{j=0}^{c}\mathcal{F}_{[i,i]}\operatorname{reg}\left(\mathcal{F}_{[0,j-1]}\varphi\right) \vee \mathcal{F}_{[i,i]}\operatorname{reg}\left(\mathcal{F}_{[j,j]}\psi\right)\right)\right).\end{aligned}$$

Since $\mathcal{F}_{[t_1,t_1]}\mathcal{F}_{[t_2,t_3]}\varphi \equiv \mathcal{F}_{[t_1+t_2,t_1+t_3]}\varphi$, we have

$$\begin{aligned}\operatorname{reg}\left(\varphi\,\mathcal{R}_{[a,b]}\gamma\right) &= \bigwedge_{i=a}^{b}\operatorname{reg}\left(\mathcal{F}_{[a,i-1]}\varphi \vee \left(\bigwedge_{j=0}^{c}\operatorname{reg}\left(\mathcal{F}_{[i,i+j-1]}\varphi\right) \vee \operatorname{reg}\left(\mathcal{F}_{[i+j,i+j]}\psi\right)\right)\right)\\ &= \bigwedge_{i=a}^{b}\bigwedge_{j=0}^{c}\operatorname{reg}\left(\mathcal{F}_{[a,i-1]}\varphi\right) \vee \operatorname{reg}\left(\mathcal{F}_{[i,i+j-1]}\varphi\right) \vee \operatorname{reg}\left(\mathcal{F}_{[i+j,i+j]}\psi\right).\end{aligned}$$

Since $\mathcal{F}_{[t_1,t_2-1]}\varphi \wedge \mathcal{F}_{[t_2,t_3]}\varphi \equiv \mathcal{F}_{[t_1,t_3]}\varphi$, we have

$$\begin{aligned}\text{reg}\left(\varphi\,\mathcal{R}_{[a,b]}\gamma\right) &= \bigwedge_{i=a}^{b}\bigwedge_{j=0}^{c}\text{reg}\left(\mathcal{F}_{[a,i+j-1]}\varphi\right) \vee \text{reg}\left(\mathcal{F}_{[i+j,i+j]}\psi\right)\\ &= \bigwedge_{i+j=a}^{b+c}\text{reg}\left(\mathcal{F}_{[a,i+j-1]}\varphi\right) \vee \text{reg}\left(\mathcal{F}_{[i+j,i+j]}\psi\right).\end{aligned}$$

Let $k = i + j$. Thus

$$\begin{aligned}\text{reg}\left(\varphi\,\mathcal{R}_{[a,b]}\gamma\right) &= \bigwedge_{k=a}^{b+c}\text{reg}\left(\mathcal{F}_{[a,k-1]}\varphi\right) \vee \text{reg}\left(\mathcal{F}_{[k,k]}\psi\right)\\ &= \text{reg}\left(\varphi\,\mathcal{R}_{[a,b+c]}\psi\right).\end{aligned}$$

Thus $\varphi\,\mathcal{R}_{[a,b]}(\varphi\,\mathcal{R}_{[0,c]}\psi) \equiv \varphi\,\mathcal{R}_{[a,b+c]}\psi$, so $\varphi\,\mathcal{B}_{[a,b+c]}\psi \equiv \varphi\,\mathcal{B}_{[a,b]}(\varphi\,\mathcal{B}_{[0,c]}\psi)$. This completes the proof.

IV Until and Release Duality Lemma

Lemma 1 **(*Until and Release Duality*).** *The definition of Release is equivalent to the dual of Until: $\varphi\mathcal{R}\psi \equiv \neg(\neg\varphi\mathcal{U}\neg\psi)$. That is to say, $\varphi\mathcal{R}_{[a,b]}\psi$ if and only if $|\pi| \leqslant a$ or $\forall s \in [a,b]$, ($\pi_s \models \psi$ or $\exists t \in [a, s-1]$, $\pi_t \models \varphi$).*

Proof.
($\Rightarrow$):

Suppose $\pi \models \varphi\mathcal{R}_{[a,b]}\psi$, so:

$$|\pi| \leqslant a \text{ or } \forall i \in [a,b], (\pi_i \models \psi) \text{ or } \exists j \in [a,b-1], (\pi_j \models \varphi \text{ and } \forall k \in [a,j]\pi_k \models \psi)$$

We proceed by cases to show that:

$$|\pi| \leqslant a \text{ or } \forall s \in [a,b], (\pi_s \models \psi \text{ or } \exists t \in [a,s-1], \pi_t \models \varphi) \tag{A1}$$

Case 0: If $|\pi| \leqslant a$, then we are immediately done. Case 1: Suppose $\forall i \in [a,b]$, $\pi_i \models \psi$.

Through re-labeling, we have $\forall s \in [a,b]$, $\pi_s \models \psi$. Then we clearly have:

$$|\pi| \leqslant a \text{ or } \forall s \in [a,b], (\pi_s \models \psi \text{ or } \exists t \in [a,s-1], \pi_t \models \varphi) \tag{A1}$$

Case 2: Suppose $\exists i \in [a,b]$, $\pi_i \not\models \psi$.

Then we must have that:

$$\exists j \in [a,b-1], (\pi_j \models \varphi \text{ and } \forall k \in [a,j]\; \pi_k \models \psi) \tag{1}$$

We want to show that $\forall s \in [a,b]$, ($\exists t \in [a,s-1]$, $\pi_t \models \varphi$).

Suppose by contradiction that:

$$\exists s \in [a,b], (\forall t \in [a,s-1], \pi_t \not\models \varphi) \tag{2}$$

Since $s \in [a, b]$ and $t \in [a, s-1]$, we have that $t \in [a, b-1]$.
Since $j \in [a, b-1]$ (from Line 1), we have that $\pi_j \not\models \varphi$ from Line 2.
However from Line 1 we have that $\pi_j \models \varphi$ and have thus derived a contradiction.
Thus, we now have that $\forall s \in [a, b]$, $(\exists t \in [a, s-1]$, $\pi_t \models \varphi)$.
From this, we clearly have that:

$$|\pi| \leqslant a \text{ or } \forall s \in [a, b], (\pi_s \models \psi \text{ or } \exists t \in [a, s-1], \pi_t \models \varphi) \tag{A1}$$

Since these 3 cases exhaustively capture all cases fo the assumption, the ($\Rightarrow$) direction is proved.
($\Leftarrow$):
Suppose that $\pi \models \neg(\neg\varphi \mathcal{U} \neg\psi)$:

$$|\pi| \leqslant a \text{ or } \forall s \in [a, b], (\pi_s \models \psi \text{ or } \exists t \in [a, s-1], \pi_t \models \varphi) \tag{1}$$

We want to show $\pi \models \varphi R_{[a,b]} \psi$, that is:

$$|\pi| \leqslant a \text{ or } \forall i \in [a, b], (\pi_i \models \psi) \text{ or } \exists j \in [a, b-1], (\pi_j \models \varphi \text{ and } \forall k \in [a, j]\ \pi_k \models \psi) \tag{B1}$$

Case 0: If $|\pi| \leqslant a$, again we are immediately done.
Case 1: Suppose $\forall s \in [a, b]$, $(\pi_s \models \psi)$.
Relabeling s to i, we now have that $\forall i \in [a, b]$, $(\pi_s \models \psi)$, which implies line B1.
Case 2: Suppose $\exists s \in [a, b], \pi_s \not\models \psi$.
By re-labeling s to i, we have that $\exists i \in [a, b], \pi_i \not\models \psi$.
Since $[a, b]$ is a finite-discrete interval, there must exist a first i_1 s.t. $\pi_{i_1} \not\models \psi$, that is:

$$\exists i_1 \in [a, b], (\pi_{i_1} \not\models \psi \text{ and } \forall k \in [a, i_1 - 1], \pi_k \models \psi) \tag{2}$$

Since (line 2) $\exists i_1 \in [a, b], \pi_{i_1} \not\models \psi$, by Line 1 we have that: $\pi_{i_1} \models \psi$ or $\exists t \in [a, i_1 - 1]$, $\pi_t \models \varphi$.
Thus, we have that:

$$\exists t \in [a, i_1 - 1], \pi_t \models \varphi \tag{3}$$

Since $[a, t] \subseteq [a, i_1 - 1]$ and $\forall k \in [a, i_1 - 1]$, $\pi_k \models \psi$, we have that:

$$\forall k \in [a, t], \pi_k \models \psi \tag{4}$$

Since $[a, t] \subseteq [a, i_1 - 1] \subseteq [a, b-1]$, we have that $t \in [a, b-1]$, so let $j := t$.
Then from lines 3 and 4, we have that:

$$\exists j \in [a, b-1], (\pi_j \models \varphi \text{ and } \forall k \in [a, j]\ \pi_k \models \psi) \tag{5}$$

From line 5, we now get:

$$|\pi| \leqslant a \text{ or } \forall i \in [a, b], (\pi_i \models \psi) \text{ or } \exists j \in [a, b-1], (\pi_j \models \varphi \text{ and } \forall k \in [a, j]\ \pi_k \models \psi) \tag{B1}$$

Again the 3 cases are exhaustive of all cases in the assumption, this we have the ($\Leftarrow$) direction.
This finishes the proof.

V Regular Expression Simplification Theorem

Theorem 7 (***Regular Expression Simplification Theorem***)**.** *Let M be a $n+1$ by n matrix, where each of the $n+1$ rows represents a regular expression of length n with commas stripped. If each column has one "1," one "0," and $n-1$ "S" characters, then the union of this set of regular expressions can be simplified to S^n, the arbitrary computation of length n.*

Proof. Assume no row is the arbitrary computation, because then the union of the set of regular expressions would trivially simplify to the arbitrary computation.
We begin by showing that there must be at least one row in the matrix M that is composed of one fixed truth value and $n-1$ 'S's. This will be of use later in the proof. Because there are n columns and 2 fixed truth values in each column, there are $2n$ fixed truth values. There are $n+1$ rows, so the average number of fixed truth values per row is strictly less than 2 since $\frac{2n}{n+1} < 2$. Thus, there must exist at least one row composed of one fixed truth value and $n-1$ 'S's.
We now proceed by induction on the length of computations, n.
Base Cases: $n = 1$ and $n = 2$.
The $n = 1$ case holds by definition:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv S.$$

For $n = 2$, we can manually verify that each possible matrix indeed satisfies the theorem. Note that because the union of regular expressions is commutative, any permutation of rows is equivalent:

$$\begin{bmatrix} 1 & S \\ S & 1 \\ 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & S \\ S & 0 \\ 1 & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & S \\ S & 0 \\ 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} 0 & S \\ S & 1 \\ 1 & 0 \end{bmatrix} \equiv SS.$$

Inductive Hypothesis:
Let $n \geqslant 2$. Assume that a matrix of regular expressions, H, with the following characteristics is equivalent to the arbitrary computation of length $n-1$: n rows, $n-1$ columns, one '1' per column, one '0' per column, and $n-2$ 'S's per column.
Inductive Step:
Consider a matrix of regular expressions, J, with the following characteristics: $n+1$ rows, n columns, one '1' per column, one '0' per column, and $n-1$ 'S's per column. We show J is equivalent to the arbitrary computation of length n.
As aforementioned, there must exist at least one row composed of one fixed truth value and $n-1$ 'S's, and the union of regular expressions is commutative. Thus, WLOG, let the first row of J, r_1, be a row with one known truth value. Suppose this known truth value is in column k, c_k, where $1 \leqslant k \leqslant n$. Assume WLOG that this value is a 0.

Matrix J can be represented as follows:

$$J = \begin{array}{cccccccc}
\mathbf{c_1} & & \mathbf{c_{k-1}} & \mathbf{c_k} & \mathbf{c_{k+1}} & & \mathbf{c_n} & \\
S & \dots & S & 0 & S & \dots & S & \mathbf{r_1} \\
 & & & S & & & & \mathbf{r_2} \\
 & & & \vdots & & & & \\
 & & & S & & & & \\
 & \vdots & & 1 & & \vdots & & \\
 & & & S & & & & \\
 & & & \vdots & & & & \\
 & & & S & & & & \mathbf{r_{n+1}}
\end{array}$$

The first row of J represents half of the regular expressions contained in the arbitrary computation. The other half would be represented by a regular expression of all 'S's, except for a '1' at the k^{th} position. Thus, if rows r_2 through r_{n+1} of J represents the other half of the arbitrary computation, then the matrix (the union of the set of the regular expressions) represents the arbitrary computation of length n. We show that this is indeed the case: Because the first row represents all the computations with '0' at the k^{th} position, every 'S' in c_k can be replaced with a '1', to avoid redundancy; the case for which each 'S_k' is '0' is a subset of r_1. Thus, matrix J can be represented as follows:

$$J = \begin{array}{cccccccc}
\mathbf{c_1} & & \mathbf{c_{k-1}} & \mathbf{c_k} & \mathbf{c_{k+1}} & & \mathbf{c_n} & \\
S & \dots & S & 0 & S & \dots & S & \mathbf{r_1} \\
 & & & 1 & & & & \mathbf{r_2} \\
 & \vdots & & \vdots & & \vdots & & \\
 & & & 1 & & & & \mathbf{r_{n+1}}
\end{array}$$

The problem reduces to showing that $J - r_1$, that is, J with the row r_1 removed, represents the other half of the arbitrary computation. Thus, let $J' = J - r_1$:

$$J' = \begin{array}{cccc}
\mathbf{c_1} & \mathbf{c_k} & \mathbf{c_n} & \\
 & 1 & & \mathbf{r_2} \\
\vdots & \vdots & \vdots & \\
 & 1 & & \mathbf{r_{n+1}}
\end{array}$$

Again, we want to show that J' represents the other half of the arbitrary computation. Recall that we specify this to be the union of regular expressions of all 'S's except for a '$1'$ at the k^{th} position. Because each row indeed contains a '$1'$ at the k^{th} position, the problem reduces to showing that $r_2 - c_k$ through $r_{n+1} - c_k$ represents the arbitrary computation, where $r_j - c_k$ is the row r_j with the k^{th} entry removed. Thus, we remove

c_k from J' and call this new matrix J'':

$$J'' = \begin{pmatrix} r_2 - c_k \\ . \\ . \\ . \\ r_{n+1} - c_k \end{pmatrix}$$

Because J'' is the result removing one row and one column from J, J'' has n rows and $n-1$ columns. In each column of J'', there remains one '1' and one '0'. Also, there are now $n-2$ 'S's in each column because r_1 was removed. Thus, J'' is equivalent to H, and is therefore equivalent to the arbitrary computation by the inductive hypothesis. Because r_1 is equivalent to half of the arbitrary computation and J' is equivalent to the other half of the arbitrary computation, J is equivalent to the arbitrary computation, as the union of r_1 and J' is equivalent to J. Therefore, by induction, the theorem holds.

VI Pseudocode for the WEST Algorithm Functions

To compute the satisfying computations of an MLTL formula, many of the functions in the WEST program require the regular expressions of the satisfying computations of the subformulas as inputs. We denote these regexes by R and T. Additionally, n will always refer to the number of propositional variables, and nnf refers to an input formula in negation normal form.

Algorithm 6 WEST Algorithm

```
Inputs: $\varphi$ - MLTL formula in NNF
$\varphi_1$ and $\varphi_2$ below are subformulas of $\varphi$
$n$ - number of propositional variables
Output: set of REGEX satisfying $\varphi$
  procedure REG(string $\varphi$, int $n$)
    if $\varphi$ is $\top$ or $\bot$ then
      return reg_prop_const($\varphi$, n)
    if $\varphi$ is $p_k$ or $\neg p_k$ then
      return reg_prop_var($\varphi$, n)
    if $\varphi = \varphi_1 \wedge \varphi_2$ then
      return set_intersect(reg($\varphi_1$), reg($\varphi_2$), n)
    if $\varphi = \varphi_1 \vee \varphi_2$ then
      return join(reg($\varphi_1$), reg($\varphi_2$), n)
    if $\varphi = \mathcal{F}_{[a,b]}\varphi_1$ then
      return reg_F(reg($\varphi_1$), a, b, n)
    if $\varphi = \mathcal{G}_{[a,b]}\varphi_1$ then
      return reg_G(reg($\varphi_1$), a, b, n)
    if $\varphi = \varphi_1\mathcal{U}_{[a,b]}\varphi_2$ then
      return reg_U(reg($\varphi_1$), reg($\varphi_2$), a, b, n)
    if $\varphi = \varphi_1\mathcal{R}_{[a,b]}\varphi_2$ then
      return reg_R(reg($\varphi_1$), reg($\varphi_2$), a, b, n)
```

Algorithm 7 Pad a set to elements of all equal length

Input: set of strings that represents a regex, number of propositional variables
Output: set of strings padded to equal length

```
procedure PAD(set $R$, int $n$)
    max_Length ← $\max_{\{r \in R\}}(r$.length())
    for ($r \in R$) do
        diff ← (max_length−$r$.length()) / $(n+1)$
        $r \leftarrow r + (,S^n)^{\text{diff}}$
    return $R$
```

Algorithm 8 Computes regex for propositional constant

Input: String that is either “true" or “false", number of propositional variables
Output: set of strings that represents the appropriate satisfying computations

```
procedure REG_PROP_CONS(string nnf, int $n$)
    if (nnf = “true" and $n \neq 0$) then return $\{S^n\}$
    else return $\{\}$
```

Algorithm 9 Output the set of computation satisfying a propositional variable

Input: String that represents a propositional variable or the negation of one, number of propositional variables
Output: set of strings that represents the appropriate satisfying computations

```
procedure REG_PROP_VAR(string nnf, int $n$)
    if (nnf = “p$k$", where $k$ is a nonnegative integer) then return $\{S^k 1 S^{n-k-1}\}$
    if (nnf = “~p$k$", where $k$ is a nonnegative integer) then return $\{S^k 0 S^{n-k-1}\}$
```

Algorithm 10 Takes the intersection of two computations

Input: Two strings representing regexes
Output: Bitwise AND of the inputted strings

```
procedure BIT_WISE_AND(string $r$, string $t$)
    ret ← “"
    for (i ∈ $[0,\ r$.length()$]$) do
        if ($r[i] \wedge t[i] =$ “”) then return “"
        else ret ← ret + $r[i] \wedge t[i]$
    return ret
```

Algorithm 11 set_intersect

Inputs: R, T - two sets of REGEX
n - number of propositional variables
Output: set of REGEX equal to $R \wedge T$

```
procedure SET_INTERSECT($R$, $T$, n)
    Pad($R$, $T$, n), return ← $\{\}$
    for $(r, t) \in R \times T$ do
        add bit_wise_and($r, t$) to ret
    return simplify(ret)
```

Algorithm 12 Takes union of two regexes (combines two sets into one)

Input: sets of strings that represent regexes, simplify boolean
Output: set of strings that represents union of inputted regexes

```
procedure JOIN(set $R$, set $T$, bool simp)
    ret ← {}
    for ($r \in R$) do add $r$ to ret
    for ($t \in T$) do add $t$ to ret
    if (simp is true) then return simplify(ret)
    else return ret
```

Algorithm 13 Computes the regex for an MLTL formula F[a,b]φ

Inputs: set of strings representing the regex for φ, interval bounds, number of propositional variables, simplify boolean
Output: set of strings that represents the appropriate satisfying computations

```
procedure REG_F(set $r_\varphi$, int $a$, int $b$, int $n$, bool simp)
    pre ← ((‘$S$’)$^n$ + ‘,’)$^a$
    comp ← $r_\varphi$
    if $a > b$ then return {}
    for ($1 \leqslant i \leqslant b - a$) do
        temp$_\varphi$ ← ((‘$S$’)$^n$ + ‘,’)$^i$ + $r_\varphi$
        comp ← join(comp, temp$_\varphi$, simp)
    return pre + comp
```

Algorithm 14 Computes the regex for an MLTL formula φU[a,b]ψ

Inputs: r_φ, r_ψ - sets of REGEX for MLTL formulas φ and ψ (after calling reg)
a, b - integers representing interval bound
n - number of propositional variables
Output: set of REGEX for $\varphi\mathcal{U}_{[a,b]}\psi$

```
procedure REG_U($r_\varphi$, $r_\psi$, $a$, $b$, $n$)
    comp ← ((‘$S$’)$^n$ + ‘,’)$^a$+ r_$\psi$
    if $a > b$ then return {}
    for ($a \leqslant i \leqslant b - 1$) do
        G1 ← reg_G($r_\varphi$, $a$, $i$, $n$)
        G2 ← reg_G($r_\psi$, $i + 1$, $i + 1$, $n$)
        temp_comp ← set_intersect(G1,       G2, n)
        comp ← join( comp, temp_comp)
    return comp
```

Algorithm 15 Computes the regex for an MLTL formula φR[a,b]ψ

Inputs: sets of strings representing the regexes for φ and ψ, interval bounds, number of propositional variables, simplify boolean
Output: set of strings that represents the appropriate satisfying computations

```
procedure REG_R(set $r_\varphi$, set $r_\psi$, int $a$, int $b$, int $n$, bool simp)
    comp ← reg_G($r_\psi$, $a$, $b$, $n$, simp)
    if $a > b$ then return $\{S^n\}$
    for ($a \leqslant i \leqslant b - 1$) do
        temp_comp ← set_intersect(reg_G($r_\psi$, $a$, $i$, $n$), reg_G($r_\varphi$, $i$, $i$, $n$), $n$, simp)
        comp ← join(comp, temp_comp, simp)
    return comp
```

Algorithm 16 Combines two strings that differ only by one character into one

Input: Two strings that represent regexes
Output: Single string that represents computations represented by both input strings or FAIL

```
procedure SIMPLIFY_STRING(string $r$, string $t$)
    if ($r$.length() $\neq$ $t$.length()) then exit
    for ($0 \leqslant i <$ $r$.length()) do
        pre_r ← $r[0, i - 1]$
        char_r ← $r[i]$
        post_r ← $r[i + 1, r$.length() $- 1]$
        pre_t ← $t[0, i - 1]$
        char_t ← $t[i]$
        post_t ← $t[i + 1, t$.length() $- 1]$
        if (pre_r = pre_t and post_r = post_t) then
            if (char_r $\neq$ char_t) then
                return pre_r +“$S$" + post_r
            else
                return $r$
    return FAIL
```

Algorithm 17 Simplifies a set of strings using simplify_string

Input: set of strings representing a regex
Output: set of strings representing a regex simplified using simplify_string

```
procedure SIMPLIFY(set $R$)
    Pad all strings in $R$ to the same length
    if ($R$.length() $\leqslant 1$) then return $R$
    $i = R$.length $- 1$
    $j = i - 1$
    START
    while ($i \geqslant 1$) do
        while ($j \geqslant 0$) do
            simplified = simplify_string($R[i]$, $R[j]$)
            if (simplified $\neq$ FAIL) then
                replace string at index $j$ with simplified
                remove string at index $i$ from $R$
                $i = R$.length() $- 1$
                $j = i - 1$
                goto START
            --$j$
        --$i$
        $j = i - 1$
    return $R$
```

Algorithm 18 Generates test suite template without propositional constants, propositional variables, or negations of propositional variables filled in

Inputs: Depth to generate, interval bounds a and b
Output: set of MLTL formulas (without propositional constants, propositional variables, or negations of propositional variables)

```
procedure GENERATE_TEST_TEMPLATE(int depth, int $a$, int $b$)
    if (depth = 0) then return $\{p, q\}$
    template $\leftarrow$ $\{\}$
    $V$ $\leftarrow$ generate_test_template(depth$-1$, $a$, $b$)
    for (string $\varphi \in V$) do
        add “G[a:b]" + $\varphi$ to template
        add “F[a:b]" + $\varphi$ to template
        for string $\psi \in V$ do
            add $\varphi$ + “U[a:b]" + $\psi$ to template
            add $\varphi$ + “R[a:b]" + $\psi$ to template
            add $\varphi$ + “$\vee$" + $\psi$ to template
            add $\varphi$ + “&" + $\psi$ to template
    return template
```

Algorithm 19 Generates a complete test suite of MLTL formulas in negation normal form up to a certain depth

```
Inputs: Depth of desired test suite template to generate, interval bounds, number of propositional variables
Output: set of MLTL formulas
  procedure GENERATE_TEST(int depth, int $a$, int $b$, int $n$)
    tests ← generate_test_template(depth, $a$, $b$)
    for string $t \in$ tests do
      for char ch $\in t$ do
        if ch $= p$ then
          $k \leftarrow$ rand()$\%n$
          if rand() $\%2 = 0$ then replace ch with p$k$
          else replace ch with ~p$k$
        else if ch $= q$ then
          if rand() $\%2 = 0$ then replace ch with T
          else replace ch with !
    return tests
```

VII State Diagram Graphs

Below we provide the state diagram graphs of the functions examined in our intelligent fuzzing. Nodes in the graph represent portions of the code without control flow statements, so a single node can represent large chunks of code. Directed edges represent branching of control flow, such as IF statements and loops. Each graph directly corresponds with the pseudocode of their corresponding function, with nodes and edges being labeled accordingly. Note that the option to run `simplify` has been omitted for clarity.

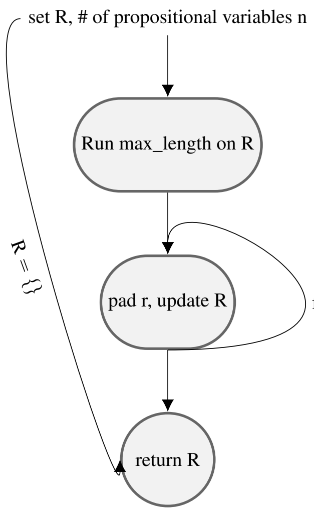

Fig. 8: Pad Function

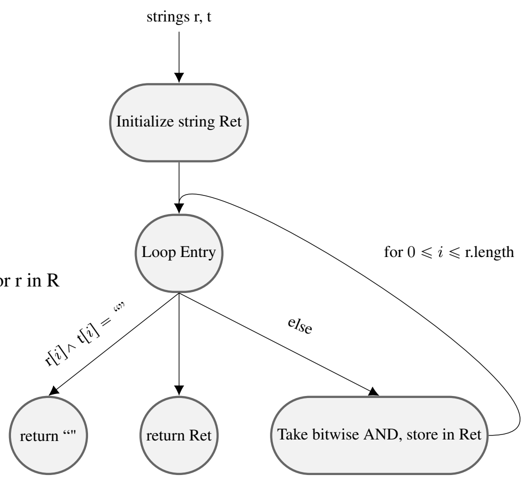

Fig. 9: Bitwise_And Function

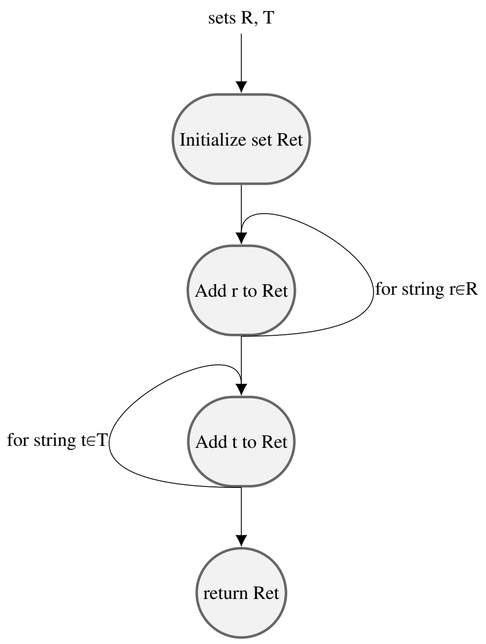

Fig. 10: Join Function

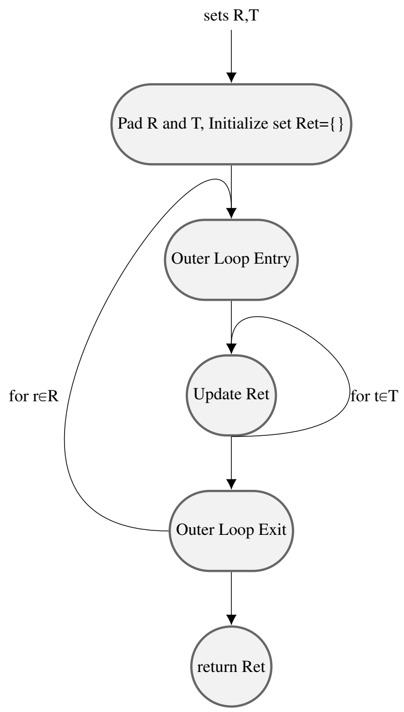

Fig. 11: Set_Intersect Function

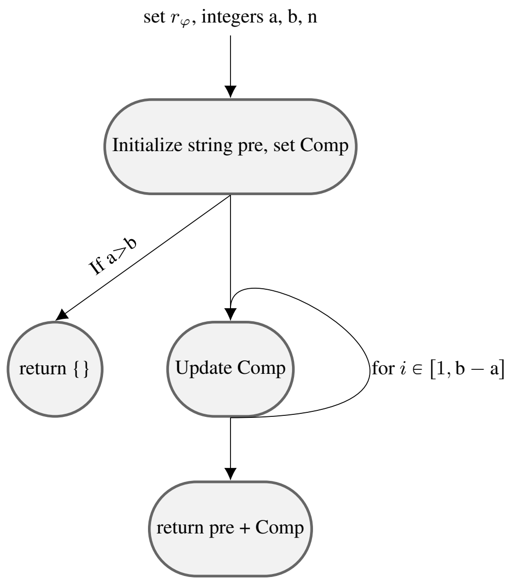
set r_φ, integers a, b, n
Initialize string pre, set Comp
If a>b
return {}
Update Comp
for $i \in [1, \mathrm{b} - \mathrm{a}]$
return pre + Comp

Fig. 12: Reg_F Function

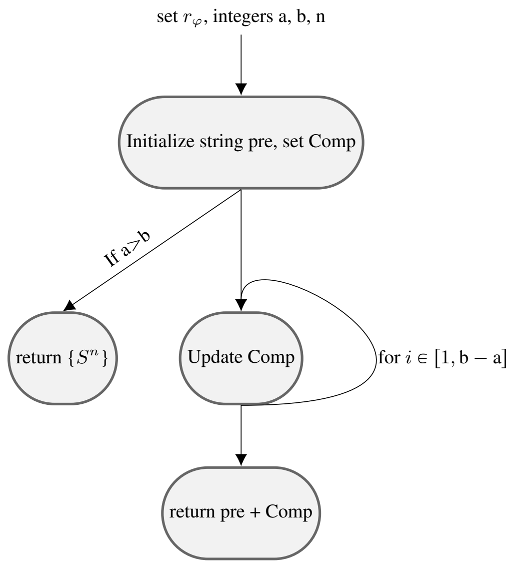
set r_φ, integers a, b, n
Initialize string pre, set Comp
If a>b
return $\{S^n\}$
Update Comp
for $i \in [1, \mathrm{b} - \mathrm{a}]$
return pre + Comp

Fig. 13: Reg_G Function

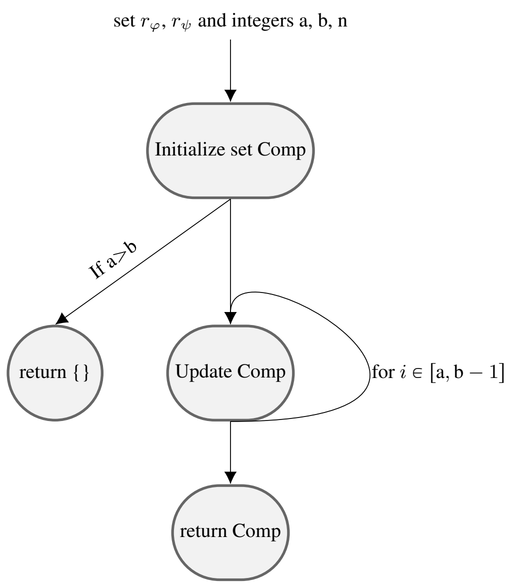

Fig. 14: Reg_U Function

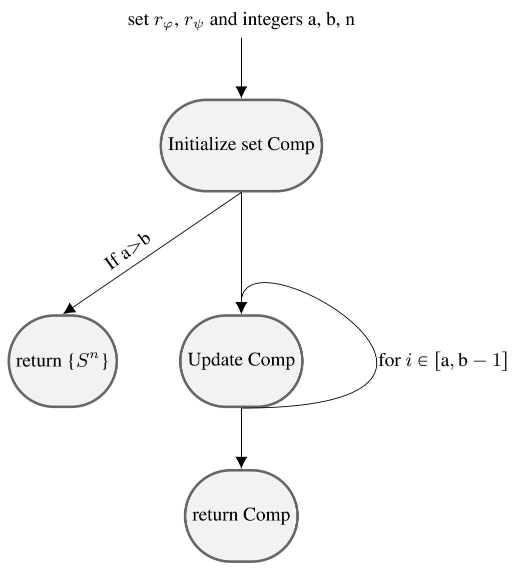

Fig. 15: Reg_R Function

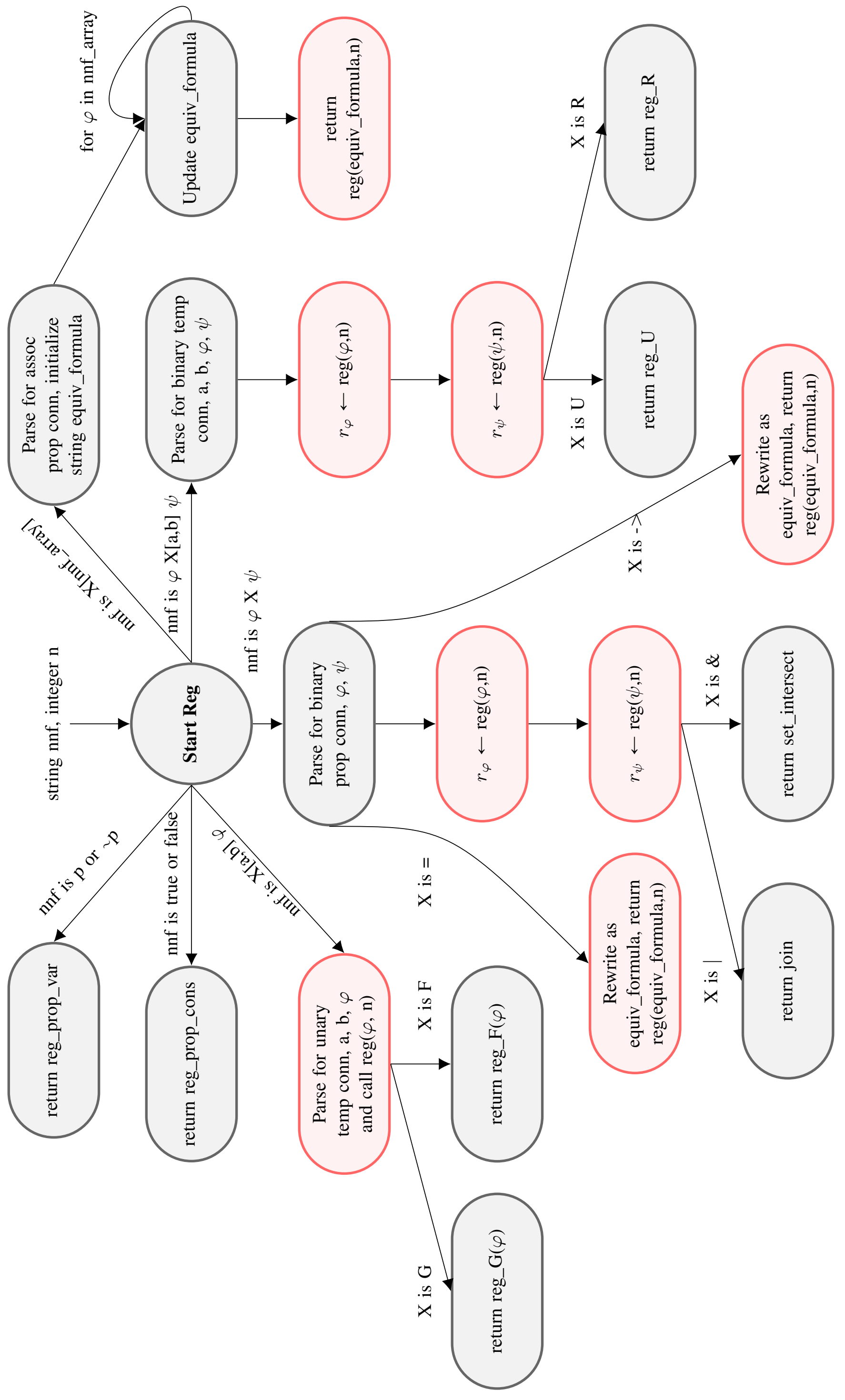
string nnf, integer n
Start Reg
nnf is p or ~p
return reg_prop_var
nnf is true or false
return reg_prop_cons
nnf is X[a,b] φ
Parse for unary temp conn, a, b, φ and call reg(φ, n)
X is F
return reg_F(φ)
X is G
return reg_G(φ)
nnf is X[nnf_array]
Parse for assoc prop conn, initialize string equiv_formula
for φ in nnf_array
Update equiv_formula
return reg(equiv_formula,n)
nnf is φ X[a,b] ψ
Parse for binary temp conn, a, b, φ, ψ
$r_\varphi \leftarrow$ reg(φ,n)
$r_\psi \leftarrow$ reg(ψ,n)
X is U
return reg_U
X is R
return reg_R
nnf is φ X ψ
Parse for binary prop conn, φ, ψ
$r_\varphi \leftarrow$ reg(φ,n)
$r_\psi \leftarrow$ reg(ψ,n)
X is &
return set_intersect
X is |
return join
X is =
Rewrite as equiv_formula, return reg(equiv_formula,n)
X is ->
Rewrite as equiv_formula, return reg(equiv_formula,n)

Fig. 16: Control Flow of Reg. Red nodes indicate a recursive call to Reg.